THE PRINCIPLES OF ALGEBRA.

AN ELEMENTARY TREATISE

ON

ALGEBRA.

BY THE

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PREFACE.

THE object of the present treatise is to give in small compass clear proofs of all the ordinary Algebraical Theorems. The best existing text-books on the subject are exhaustive. Explanations of the most elementary facts and elaborate discussions of most minute and subtle points are alike to be found in them. Consequently, the student of ordinary capacity who is preparing for Cambridge, Woolwich, or the Indian Civil Service, finds what he wants imbedded in a vast mass of what is not necessary for him to read. From this objection the present treatise is free. Moreover, the proofs here given are as far as possible independent, and *in no single instance have two demonstrations of the same theorem been offered.* The writer's experience in tuition has convinced him that it is most inexpedient to present such an alternative to most learners. It generally leads to their confusing the two proofs, and being unable to reproduce either.

The Examples which will be found at the end partake more of the character of problems, than those which are

usually given in text-books, and are selected chiefly from Cambridge Examination Papers.

My thanks are due to the Rev. J. R. Lunn, B.D., late Fellow and Lecturer of St. John's, and to Mr. C. Body, B.A., Scholar of St. John's, for much valuable assistance; also to the Rev. Percival Frost, M.A., Lecturer at King's, for kindly allowing me to use his methods on several occasions, notably in Sections 68 and 70.

J. H. R.

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CONTENTS.

Algebra.

1. **A Term** is a quantity which serves to make up an Algebraical expression by way of addition or subtraction.

A Factor is a quantity which serves to make up an Algebraical expression by way of multiplication.

Thus $2a^2+3ab+5b^2$ consists of three terms, each of which consists of several factors.

The first term	$2a^2$	consists of three factors		$2.a.a.$
The second	$3ab$	"	"	$3.a.b.$
The third	$5b^2$	"	"	$5.b.b.$

A simple factor consists of only one term. **A compound factor** of two or more terms.

Thus $2a(b+c)$ consists of two factors.

A simple factor $2a$ and a compound factor $(b+c)$.

In N^r and D^r of a fraction we cannot cancel *terms:* $\frac{a+b}{a+c}$ is not $\frac{b}{c}$. We may cancel *factors:* $\frac{ab}{ac}$ *is* $\frac{b}{c}$ for we *may divide* N^r and D^r by the same quantity "a."

2. **Def.** The repetition of a factor is denoted by a small figure placed above it to the right, indicating the number of times it is repeated.

Thus $a \times a \times a \times a$ is denoted by a^4,

$a \times a \times a \times \;.\;.\;.\;.\;.$ to 20 factors, by a^{20};

or generally $a \times a \times a \times \;.\;.\;.$ to m factors by a^m.

The small figure is called an index.

3. **The whole of Algebra is founded on three Laws—**

(i) **The Index Law** . . . $a^m \times a^n = a^{m+n}$.

(ii) **The Commutative Law** . $ab = ba$.

(iii) **The Distributive Law** . $a(b \pm c) = ab \pm ac$.

4. **To prove $a^m \times a^n = a^{m+n}$, m and n being positive integers.**

By Def. $a^m = a.a.a \ldots$ to m factors,

and $a^n = a.a.a \ldots$ to n factors;

$\therefore a^m \times a^n = a.a.a \ldots$ to m factors, $\times\, a.a.a \ldots$ to n factors,

$= a.a.a \ldots$ to $(m+n)$ factors.

$= a^{m+n}$ by def.

This translated into words is "In Multiplication add the Indices."

5. Arithmetical and Symbolical Algebra differ as follows:—

(i) **In the Symbols used**, which in Arithmetical Algebra denote positive integers, but in Symbolical Algebra quantities of any kind whatever.

(ii) **In the Signs + and −**, which in Arithmetical Algebra denote the operations of Addition and Subtraction, but in Symbolical Algebra the *affections* of the symbols to which they are applied.

(iii) **In the grounds on which they rest.** The laws in Arithmetical Algebra are deduced from the nature of numbers (as in 4), but in Symbolical Algebra these laws of combination are assumed to be *universally* true, and the meaning of the signs and symbols is interpreted.

6. **To interpret the meaning of a^m on the assumption $a^m \times a^n = a^{m+n}$.**

(i) Let m be a + ve integer.

Then $a^m = a^{1+1+1+\ldots \text{ to } m \text{ terms}} = a.a.a \ldots$ to m factors.

(ii) Let m be a $+$ve fraction $=\frac{r}{s}\cdot$

Then $a^{\frac{r}{s}}\times a^{\frac{r}{s}}\times a^{\frac{r}{s}}\ldots$ to s factors $=a^{\frac{r}{s}+\frac{r}{s}+\ldots \text{ to } s \text{ terms}}=a^r$;

$$\therefore\ a^{\frac{r}{s}}= \text{the } s\text{th root of } a^r=\sqrt[s]{a^r}.$$

(iii) Let m be negative $=-n$.

Then $a^{m+n}\times a^{-n}=a^m=a^m.a^n.\frac{1}{a^n}$;

$$\therefore\ a^{-n}=\frac{1}{a^n}\cdot$$

(iv) Let $m=0$.

Then $a^m\times a^0=a^{m+0}=a^m$;

$$\therefore\ a^0=1.$$

COR. $a^m\div a^n=a^m\times\frac{1}{a^n}$, which by (iii),

$=a^m\times a^{-n}=a^{m-n}$.

7. **To prove that** $(a^m)^n=a^{mn}$.

(i) Let m be a $+$ve integer.

Then $(a^m)^n=a^m\times a^m\times a^m\times\ \ldots$ to n factors,

$=a^{m+m+m+\ \ldots \text{ to } n \text{ terms}}$,

$=a^{nm}=a^{mn}$.

(ii) Let $n=\frac{r}{s}$, a fraction.

Then $(a^m)^{\frac{r}{s}}=\sqrt[s]{(a^m)^r}=\sqrt[s]{a^{mr}}=a^{\frac{mr}{s}}$.

(iii) Let n be negative $=-r$.

Then $(a^m)^{-r}=\frac{1}{(a^m)^r}=\frac{1}{a^{mr}}=a^{-mr}$.

8. Hence the rules of indices are—

In multiplication add the indices. Thus $a^3\times a^4=a^7$.
In division subtract them, ... $a^6\div a^2=a^4$.
In raising to a power multiply them, . . $(a^2)^3=a^6$.
In taking a root divide them, ... $\sqrt[3]{a^6}=a^2$.

9. Proof of the Commutative Law :—

$$
\begin{aligned}
ba = \; & a+a+a+ \;\ldots \text{ to } b \text{ terms,} \\
& 1+1+1+ \;\ldots \text{ to } a \text{ terms,} \\
& +1+1+1+ \;\ldots \text{ to } a \text{ terms,} \\
& +1+1+1+ \;\ldots \text{ to } a \text{ terms,} \\
& \text{the number of rows being } b, \\
= \; & b+b+b+ \;\ldots\ldots \text{ to } a \text{ terms,} \\
= \; & ab.
\end{aligned}
$$

10. Proof of the Distributive Law :—

$$m(a-b)=ma-mb.$$

For $m(a-b)=(a-b)m$ and so denotes $(a-b)$ groups of m each. Hence if we had a such groups we should have b groups too many, *i.e.* if we had am things we should have bm things too many. We should therefore have to subtract these bm things;

$$\therefore\; m(a-b)=am-bm=ma-mb.$$

NOTE.—*ab means a times b.*

Similarly it may be proved that $m(a+b)=ma+mb$.

11. Rule of Signs in Multiplication.

Since $m(a-b)=ma-mb$, let $m=c-d$;

$$
\begin{aligned}
\therefore\; (c-d)(a-b) &= (c-d)a-(c-d)b, \\
&= (ca-da)-(cb-bd).
\end{aligned}
$$

Hence from the remainder found by subtracting da from ca, we must take the remainder found by subtracting bd from cb. Thus, if from $ca-da$ we took cb, we should have taken bd too much. We should therefore now have to add bd;

$$\therefore\; (c-d)(a-b)=ca-da-cb+bd.$$

Whence it appears that in multiplying $a-b$ by $c-d$, we multiply each term in the one by each term in the other, and when the factors multiplied have like signs, we prefix the sign $+$, and when unlike signs the sign $-$. *This law is the rule of signs.*

Note.—Such cases as $(+x)\times(-y)$ and $(-x)\times(-y)$ can

occur in Arithmetical Algebra only as single terms in combination with some others. In Symbolical Algebra this is not so; *e.g.* in the above let $a=0$ and $c=0$, and then $(-b)\times(-d)=+bd$.

ADDITION AND SUBTRACTION.

12. We accept it as axiomatic that we cannot collect and express in one term any but like quantities.

For instance we cannot say 2 a's and 3 b's are 5 of any unit whatever. In adding them all we can do is to couple them by the sign $+$, thus $2a+3b$.

With like quantities the case is different. Their sum can be expressed in one term. Thus $2a$ and $3a$ are, when added, $5a$. $2a^2b$ and $3a^2b$ are $5a^2b$, and so on.

This same rule evidently holds with respect to subtraction.

13. **Rule of Signs in Subtraction.**

"Change signs in the subtrahend and add."

Proof.—Let the minuend be a and the subtrahend $b-c$, and we are required to find the result when $b-c$ is taken from a. Consider as an approximate result $a-b$. This is clearly too little, for we have taken from a *the whole of* b, instead of a quantity c less than b. Therefore we have subtracted c too much; therefore our result is c too little. Hence the true result is $a-b+c$, which is what would have been obtained by changing the signs in the subtrahend and adding.

Brackets are merely cases of addition or subtraction. A bracket preceded by $+$ denotes the *addition* of the enclosed quantities to the former term. Consequently there will be no change of signs on removing the brackets. Thus

$$a+(b-c)=a+b-c.$$

A bracket preceded by $-$ denote the *subtraction* of the enclosed quantities from the former term. Consequently the signs of the enclosed quantities must be changed on the removal of the brackets.

Thus $a-(b-c)=a-b+c$.

DIVISION.

14. Observe $x^n \pm y^n$ is exactly divisible by $x \pm y$ when n is odd, and the signs in Dividend and Divisor the same, or when n is even and the sign in the Dividend is $-$. All other cases will leave a remainder.

The signs in the quotient will be all $+$ if we divide by $x-y$, and alternate if we divide by $x+y$.

The *coefficients* are all unity.

The *terms* are x^{n-1}, $x^{n-2}y$, $x^{n-3}y^2$, etc., where the indices of x decrease regularly by unity, and the indices of y similarly increase.

An example or two will make this plain.

(i) $\dfrac{x^5-y^5}{x-y}$.

This is divisible, for the indices are odd and the signs alike.

The terms are x^4, x^3y, x^2y^2, xy^3, y^4.

The signs are all $+$.

Hence the quotient is $x^4+x^3y+x^2y^2+xy^3+y^4$.

(ii) $$\frac{a^3+8b^3}{a+(2b)}=\frac{a^3+(2b)^3}{(a)+(2b)}=a^2-a(2b)+(2b)^2$$
$$=a^2-2ab+4b^2.$$

15. A knowledge of the preceding section is of great use in factorizing. If we know that a certain quantity, 15 say, is divisible by 5, it follows that 5 is one factor and the quotient (3) the other factor.

So if we have x^3+y^3 we know that $(x+y)$ is contained in it (x^2-xy+y^2) times;

$$\therefore\ x^3+y^3=(x+y)(x^2-xy+y^2),$$

$$\text{so}\quad x^3-y^3=(x-y)(x^2+xy+y^2).$$

It should also be remembered that

$$(x+y)^2=x^2+2xy+y^2$$
$$(x-y)^2=x^2-2xy+y^2$$
$$(x+y)(x-y)=x^2-y^2$$
$$(x+y)^3=x^3+y^3+3xy(x+y)$$
$$(x-y)^3=x^3-y^3-3xy(x-y).$$

16. Quantities are said to be homogeneous when the sum of the indices is the same in each term.

Thus $x^3+3x^2y+3xy^2+y^3$ is homogeneous, the sum of the indices in each term being 3.

Quantities are said to be symmetrical when all the symbols are similarly involved in them.

Thus x^3+x+y^3+y is symmetrical.

If A be a homogeneous expression of m dimensions and B another of n dimensions, AB is homogeneous in $(m+n)$ dimensions; $\frac{A}{B}$ is homogeneous in $(m-n)$ dimensions; A^p is homogeneous in mp dimensions, and $\sqrt[p]{A}$ is homogeneous in $\frac{m}{p}$ dimensions. We can often, by remembering this and noticing where the homogeneity is broken, detect errors in working.

17. **Horner's Rule.**—*If $f(x)$ be an expression of the form $p_nx^n+p_{n-1}x^{n-1}+\ .\ .\ .\ +p_1x+p_0$, the remainder when it is divided by $x-a$, is $f(a)$, (all the symbols denoting finite quantities).*

Let R denote this remainder, which cannot contain x or the division might be carried further. Let Q denote the quotient which is of the form $q_{n-1}x^{n-1}+q_{n-2}x^{n-2}+\ .\ .\ .\ +q_0$, and Q' the resulting quantity when a is written instead of x in Q. Then since the symbols all denote finite quantities Q' is finite.

Now since universally Dividend = (Divisor × Quotient) + Remainder;

$$\therefore f(x)=Q(x-a)+R,$$

and this is true for all values of x. Therefore let $x=a$.

Hence $f(a)=Q'(0)+R$, and Q' is finite;

$$\therefore R=f(a).$$

So if $f(x)$ be divided by $(x+a)$ the remainder is $f(-a)$.

Note.—$f(x)$ is a rational integral algebraical function of x, and to say that its remainder is $f(a)$ when it is divided by $x-a$ is equivalent to saying that the remainder is the same as the dividend, *only with a written instead of x.*

Thus the remainder when x^2-3x+2 is divided by $x-5$ is $5^2-3.5+2=12$, which we obtain otherwise by actual division:

$$\begin{array}{r|l}
x-5 & x^2-3x+2\ (x+2 \\
 & \underline{x^2-5x} \\
 & \quad 2x+2 \\
 & \quad \underline{2x-10} \\
 & \qquad \underline{12.}
\end{array}$$

Take another example.

By Horner's rule the remainder when x^3-px^2+qx-r is divided by $x-a$ is a^3-pa^2+qa-r.

Found by actual division thus:

$$\begin{array}{r|l}
x-a & x^3-px^2+qx-r\ (x^2-(p-a)x-(ap-a^2-q) \\
 & \underline{x^3-ax^2} \\
 & -(p-a)x^2+qx \\
 & \underline{-(p-a)x^2+(ap-a^2)x} \\
 & \qquad -(ap-a^2-q)x-r \\
 & \qquad \underline{-(ap-a^2-q)x+(a^2p-a^3-qa)} \\
 & \qquad\qquad \underline{a^3-pa^2+qa-r.}
\end{array}$$

The student will see subsequently the utility of thus quickly writing down a remainder.

18. A measure of a quantity is that which is contained in it exactly.

A common measure of two or more quantities is one which is contained in *each* exactly.

The greatest common measure (G.C.M.) of two or more quantities, is the *greatest* quantity contained in each exactly.

In algebra the term *highest common divisor* is used sometimes in preference to G.C.M., because we seek in this subject for that common divisor which is highest in dimensions, not that which is greatest in magnitude: and it may happen that these are opposed.

Thus $(x-1)$ and $(x-1)^3$ are alike common divisors of $(x-1)^3$ and $(x-1)^4$, but $(x-1)^3$ is the H.C.D.

Now if $x=\frac{3}{2}$ say, $(x-1)=\frac{1}{2}$, and $(x-1)^3=\frac{1}{8}$, so that the H.C.D. is *less* in magnitude than the other divisor $(x-1)$. It is therefore *not* the G.C.M. Although for this reason it would be better to use the term H.C.D. in algebra, the term G.C.M. is still generally employed.

19. Proof of the rule for finding the G.C.M. of two quantities.

(i) **Lemma.**—Any quantity x which measures a and b also measures $ma \pm nb$.

For let x be contained in a, p times, and in b, q times;

$$\therefore\ a=px, \text{ and } b=qx;$$
$$\therefore\ ma=mpx, \text{ and } nb=nqx;$$
$$\therefore\ ma\pm nb=(mp\pm np)x;$$

$\therefore$ x is contained in $ma\pm nb$, viz., $(mp\pm nq)$ times.

(ii) To find the G.C.M. of a and b when they contain no simple factors.

$$\begin{array}{l} b\,)\,a\,(\,p \\ \quad\ \underline{pb} \\ \quad\ \ c\,)\,b\,(\,q \\ \qquad\ \ \underline{qc} \\ \qquad\ \ \ d\,)\,c\,(\,r \\ \qquad\qquad \underline{rd} \\ \qquad\qquad \underline{\cdots} \end{array}$$

Let b be not of higher dimensions than a, and divide as in the subjoined scheme.

Then any quantity which measures a and b also measures $a-pb$, that is c (by the Lemma); and it measures b.

$\therefore$ any quantity which measures a and b also measures b and c.

Again, any quantity which measures b and c also measures $c+pb$, that is a; and it measures b.

$\therefore$ any quantity which measures b and c also measures a and b.

$\therefore$ a and b and b and c have precisely the same common measures.

Similarly c and d have precisely the same common measures as b and c, and $\therefore$ as a and b.

Now if d is exactly contained in c, then, since no quantity greater than d measures d, d is the G.C.M. of c and d, and therefore of a and b.

(iii) To find the G.C.M. of α and β which contain simple factors. Let F be a simple factor contained in α, and a the resulting quantity when this factor is struck out—f a simple factor in β, and b the resulting quantity when this factor is struck out;

$$\therefore\ \alpha=Fa, \text{ and } \beta=fb.$$

Now the common simple factor, if any exist, can be found by inspection from F and f; and the common compound factor, if any exist, from a and b by the ordinary method. The product of the common simple factor and the common compound factor is obviously the G.C.M. sought.

(iv) If, in course of working, a simple factor occur in a remainder, we may strike it out and take no further notice of it.

Let the remainder c contain a simple factor f, and c' be the resulting quantity when this is struck out.

$$\begin{array}{l} b\,)\,a\,(\,p \\ \underline{\;pb\;} \\ \quad c=fc'. \end{array}$$

Then the compound factor we are in search of is in c; $\therefore$ in fc', but it cannot be in f a simple factor, $\therefore$ *it is in* c'; $\therefore$ we may work with c' instead of c.

(v) If the first term of a be not divisible by the first term of

b we may multiply a by some simple factor f which will make it so.

For since f contains only simple factors it is clear that the compound factor common to fa and b must be identical with that common to a and b.

20. To find the G.C.M. of three quantities a, b, and c.

Let x be the G.C.M. of a and b.

$\therefore$ x contains all the factors common to a and b.

$\therefore$ the factor common to a, b, and c, must be common to x and c.

$\therefore$ the G.C.M. of x and c will be the G.C.M. of a, b, and c.

21. A common multiple of two or more quantities is any quantity which exactly contains each of them; and their L.C.M. is the *least* quantity which so contains them.

The L.C.M. is generally in practice found by inspection. All we have to do is to write down all the quantities in their most fully factorized forms. The expression which involves every factor that appears is the L.C.M.

Thus to find the L.C.M. of $ab(a^3+b^3)$, $a^2(a+b)^2$, $b^2(a-b)$,

$$ab(a^3+b^3)=ab(a+b)(a^2-ab+b^2),$$
$$a^2(a+b)^2=a^2(a+b)^2,$$
$$b^2(a-b)=b^2(a-b);$$
$$\therefore \text{ the L.C.M. } =a^2b^2(a+b)^2(a-b)(a^2-ab+b^2).$$

Note.—Remember to write the *highest* power of any factor, for *that* has to be contained in the L.C.M.

22. If x be the G.C.M. of a, and b their L.C.M. $=\dfrac{a}{x}\times b$.

For let x be contained in a, p times, and in b, q times;

$\therefore$ $a=px$ and $b=qx$, and p and q are prime to one another;

$$\therefore \text{ the L.C.M.} =pqx=\frac{px}{x}\cdot qx=\frac{a}{x}\times b.$$

23. The method of working the following example should be observed.

If $(x-a)$ be the G.C.M. of x^2+px+q and x^2+rx+s, then will $a=\dfrac{s-q}{p-r}$.

For the remainders, when these two quantities are divided by $x-a$ are a^2+pa+q and a^2+ra+s (Horner's rule).

But since $x-a$ *measures* the given quantities these remainders must severally $=0$;

$$\therefore\ a^2+pa+q=0,$$
$$\text{and}\ a^2+ra+s=0;$$
$$\therefore\ (p-r)a+(q-s)=0;$$
$$\therefore\ (p-r)a=s-q;$$
$$\therefore\ a=\frac{s-q}{p-r}.$$

24. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$ then any one of these fractions

$$=\frac{a\pm c\pm e}{b\pm d\pm f}.$$

Let $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=x$;

$$\therefore\ a=bx,\quad \pm c=\pm dx,\quad \pm e=\pm fx;$$
$$\therefore\ a\pm c\pm e=(b\pm d\pm f)x;$$
$$\therefore\ x=\frac{a\pm c\pm e}{b\pm d\pm f}.$$

The student will easily see that either fraction may be proved

$$=\frac{ma\pm nc\pm pe}{mb\pm nd\pm pf}.$$

25. If $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

For $\dfrac{a}{b}+1=\dfrac{c}{d}+1$. $\quad\therefore\ \dfrac{a+b}{b}=\dfrac{c+d}{d}$,

and $\frac{a}{b}-1=\frac{c}{d}-1$; $\quad\therefore\ \frac{a-b}{b}=\frac{c-d}{d}$,

whence by division $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

This result is extremely useful in dealing with two fractions in equations, etc.

26. **The following methods of simplification should be carefully noticed.**

(i) Prove

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a).$$

For since $b=a$ makes the D^r vanish,

$(a-b)$ is a factor of the D^r.

So $(b-c)$ and $(c-a)$ are factors of the D^r,

and $(a-b)$, $(b-c)$, $(c-a)$, are prime to each other;

$\therefore\ D^r=k(a-b)(b-c)(c-a)$.

Now from a comparison of dimensions it is plain that k does not involve a, b, or c.

Therefore since the N^r is found from the D^r by writing a^2, b^2, c^2, for a, b, c, respectively.

$\therefore$ the $N^r=k(a^2-b^2)(b^2-c^2)(c^2-a^2)$ *k being unchanged,*

$$\therefore \text{ the fraction } =\frac{k(a^2-b^2)(b^2-c^2)(c^2-a^2)}{k(a-b)(b-c)(c-a)}$$

$$=(a+b)(b+c)(c+a). \qquad \text{Q.E.D.}$$

(ii) Prove $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$.

First method.

Since $b=a$ makes the left-hand side vanish.

$(a-b)$ is a factor of it.

So also are $(b-c)$ and $(c-a)$;

$\therefore$ the left-hand side $=k(a-b)(b-c)(c-a)$.

Now the value of k is clearly independent of abc;

$$\therefore \text{ let } a=3,\ b=2,\ c=1;$$

$$\therefore 1+1+(-2)^3=k.1.1(-2);$$

$$\therefore 2k=6;\ \therefore k=3;$$

$$\therefore \text{the left-hand side } =3(a-b)(b-c)(c-a).$$

Second method.

Let $(a-b)=x$, $(b-c)=y$, and $\therefore (c-a)=-(x+y)$.

Hence left-hand side $=x^3+y^3-(x+y)^3$

$$=-3xy(x+y)$$

$$=3(a-b)(b-c)(c-a).$$

It is important to remember the following factorization :—

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

as it frequently occurs in problems.

27. When an expression is raised to any power the process is called *Involution*, and when a root is taken the process is called *Evolution*.

Since any even number of like signs multiplied together produce $+$, it is clear that we cannot have an *even* root of a negative quantity, and that in taking an even root of a $+$ ve quantity, we must prefix a double sign $\pm$.

Thus $\sqrt{-4}$ is purely imaginary. Neither -2 nor $+2$ when squared will produce -4.

But $\sqrt{4}=\pm 2$ for $+2$ or -2 when squared alike produce 4.

It is obvious that an *odd* root of a negative quantity is *not* imaginary, for an odd number of $-$ signs produces $-$. Thus $\sqrt[3]{-8}=-2$.

28. The following is a quick way of squaring a multinomial :—

Rule.—Let A be the quantity to be squared.

Beneath A write down its double.

Square the first term in A, and multiply the first term by all the doubles which follow it.

Square the second term in A, and multiply the second term by all the doubles which follow it.

Deal similarly with each of the quantities in A, and finally add.

Example.

$$\begin{array}{l} 1-x+x^2 \\ \underline{2-2x+2x^2} \\ 1-2x+2x^2 \\ \qquad\quad\ \ \underline{x^2-2x^3+x^4} \\ \underline{1-2x+3x^2-2x^3+x^4} \end{array}$$

Reason for the rule,

$$\begin{aligned}(a+b+c+\ldots)^2 &= a^2+2a(b+c+\ldots)+(b+c+\ldots)^2 \\ &= a^2+a(2b+2c+\ldots)+b^2+2b(c+\ldots)+(c+\ldots)^2 \\ &= a^2+a(2b+2c+\ldots) \\ &\quad\ +b^2+b(2c+\ldots) \\ &\qquad\ +c^2+\text{etc.}\end{aligned}$$

29. We shall presume on the student's knowledge of the ordinary methods of extracting square and cube roots. As in division, however, he must be careful before working to arrange the terms so that the indices of some chosen symbol may gradually increase or gradually decrease. He should remember also that a numerical coefficient supposes a literal coefficient with index 0.

Thus $a^2+2+a^{-2}=a^2+2a^0+a^{-2}$.

For instance, if we are required to find the square root of $\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{2x}{y}+\frac{2y}{x}+3$, we must first arrange with reference to one of the symbols, say x, and write

$$\frac{x^2}{y^2}+\frac{2x}{y}+3+\frac{2y}{x}+\frac{y^2}{x^2},$$

where the indices of x, viz., 2, 1, 0, -1, -2, decrease regularly.

The example may be worked as follows :—

$$
\begin{array}{l|l}
 & \frac{x^2}{y^2}+\frac{2x}{y}+3+\frac{2y}{x}+\frac{y^2}{x^2}\left(\frac{x}{y}+1+\frac{y}{x}\right. \\
 & \frac{x^2}{y^2} \\
\hline
\frac{2x}{y}+1 & \frac{2x}{y}+3 \\
 & \frac{2x}{y}+1 \\
\hline
\frac{2x}{y}+2+\frac{y}{x} & 2+\frac{2y}{x}+\frac{y^2}{x^2} \\
 & 2+\frac{2y}{x}+\frac{y^2}{x^2} \\
\hline
\end{array}
$$

Another illustration will be useful.

Find the square root of $\sqrt[4]{x^3}+\sqrt[4]{y^3}-2\sqrt[8]{x^3y^3}$.

(i) *Convert into fractional indices.*

(ii) *Arrange*

$$
\begin{array}{l|l}
 & x^{\frac{3}{4}}-2x^{\frac{3}{8}}y^{\frac{3}{8}}+y^{\frac{3}{4}}\left(x^{\frac{3}{8}}-y^{\frac{3}{8}}\right. \\
 & x^{\frac{3}{4}} \\
\hline
2x^{\frac{3}{8}}-y^{\frac{3}{8}} & -2x^{\frac{3}{8}}y^{\frac{3}{8}}+y^{\frac{3}{4}} \\
 & -2x^{\frac{3}{8}}y^{\frac{3}{8}}+y^{\frac{3}{4}} \\
\hline
\end{array}
$$

Thus *conversion into fractional* indices and *arrangements* are the two important things to bear in mind in working roots of this kind.

30. We have seen that quantities like $\sqrt{-4}$ are purely imaginary—that no *approximation* to their value can be found.

With quantities like $\sqrt{2}$ the case is different. These are called *Surds*, and though their value cannot be found exactly, still approximations may be made, approaching nearer and nearer to the true value, as we go on extracting the root to more decimal places.

It may be well to remember that

$$\sqrt{2}=1.414\ldots$$
$$\sqrt{3}=1.732\ldots$$

31. A quantity like $\sqrt{32}$ may be obviously written

$$\sqrt{16.2}=\sqrt{16}.\sqrt{2}=4\sqrt{2}.$$

In this case $\sqrt{32}$ is called an entire surd and $4\sqrt{2}$ is that entire surd reduced to its simplest form.

The number 4 is the *rational factor*, and $\sqrt{2}$ the *surd factor*.

32. **Similar Surds** are those which have or may be made to have the same surd factor.

Thus $\sqrt{12}$ and $\sqrt{75}$ are similar surds, for they may be written $2\sqrt{3}$ and $5\sqrt{3}$ respectively, when each has the same surd factor, viz., $\sqrt{3}$.

33. Similar surds (as like quantities) can evidently be added and expressed in one term. Thus $2\sqrt{3}+5\sqrt{3}=7\sqrt{3}$.

Dissimilar surds, however, cannot be so added and expressed in one term; $2\sqrt{3}$ and $5\sqrt{7}$ when added must be simply coupled together by the sign $+$, thus

$$2\sqrt{3}+5\sqrt{7}.$$

The same applies evidently to subtraction.

34. *Surds should never be left in Denominators.*

A single example will show why not.

Suppose we have to find $\frac{\sqrt{3}}{\sqrt{2}}$ to 7 places of decimals.

Then three processes are involved—

(i) To find $\sqrt{3}$.
(ii) To find $\sqrt{2}$.
(iii) To divide the former by the latter.

Now dividend and divisor each involve several places of decimals, and the division will be lengthy.

But if we multiply N^r and D^r by $\sqrt{2}$, so obtaining $\frac{\sqrt{6}}{2}$, only two processes are involved—

(i) To find $\sqrt{6}$.

(ii) To divide the result by 2.

Thus obtaining the required result with much less labour and with greater accuracy, because the divisor is not merely approximate.

35. It becomes then an important question what multiplier we are to employ in order to convert into rational quantities any surd quantities which may occur in denominators.

It will depend entirely on the form of these surds.

36. If the denominator consist of a single surd factor as in $\sqrt[n]{\frac{a}{b}}$ we have obviously to multiply N^r and D^r by the least power of b, which will make the index of the D^r divisible by n.

Thus $\sqrt[3]{\frac{3}{2}}=\sqrt[3]{\frac{3.2^2}{2^3}}=\frac{1}{2}\sqrt[3]{12}.$

$$\sqrt[5]{\frac{3}{2^{11}}}=\sqrt[5]{\frac{3.2^4}{2^{15}}}=\frac{1}{2^3}\sqrt[5]{3.16}=\frac{1}{8}\sqrt[5]{48}.$$

37. If the D^r be of the form $\sqrt{a}\pm\sqrt{b}$ it will evidently be rationalized by multiplying N^r and D^r by $\sqrt{a}\mp\sqrt{b}$. In other words the rationalizing multiplier is the D^r *with the sign changed.* The same clearly holds when the D^r is of the form $a\pm\sqrt{b}$.

Thus

$$\frac{1}{\sqrt{3}-\sqrt{2}}=\frac{\sqrt{3}+\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}=\frac{\sqrt{3}+\sqrt{2}}{3-2}=\sqrt{3}+\sqrt{2}.$$

38. If the D^r in the form $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

An example worked will sufficiently explain the method.

$$\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}=\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})}$$

$$=\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{(a+2\sqrt{ab}+b)-c}=\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{(a+b-c)+2\sqrt{ab}}$$

$$=\frac{(\sqrt{a}+\sqrt{b}-\sqrt{c})(a+b-c-2\sqrt{ab})}{(a+b-c)^2-4ab}.$$

39. If the D^r be of the form $a^{\frac{1}{3}}-b^{\frac{1}{2}}$.

The minimum rational product is clearly

$$a^{\frac{6}{3}}-b^{\frac{6}{2}}=a^2-b^3;$$

$\therefore$ the rationalizing multiplier is

$$a^{\frac{5}{3}}+a^{\frac{4}{3}}b^{\frac{1}{2}}+ab+a^{\frac{2}{3}}b^{\frac{3}{2}}+a^{\frac{1}{3}}b^2+b^{\frac{5}{2}}.$$

Take a numerical example,

$(2^{\frac{1}{3}}-3^{\frac{1}{4}})\times m$ is rational. Find the least value of m.

Here the minimum rational product being 2^4-3^3, we have

$$m=\frac{2^4-3^3}{2^{\frac{1}{3}}-3^{\frac{1}{4}}}=2^{\frac{11}{3}}+2^{\frac{10}{3}}3^{\frac{1}{4}}+\ \ldots\ +3^{\frac{11}{4}}.$$

40. *The product of two dissimilar surds cannot be rational.*

If possible let $\sqrt{x}.\sqrt{y}=m$, a rational quantity, $\sqrt{x}$ and $\sqrt{y}$ being dissimilar surds;

$$\therefore\ xy=m^2;$$

$$\therefore\ y=\frac{m^2}{x}=\frac{m^2}{x^2}\cdot x;$$

$$\therefore\ \sqrt{y}=\frac{m}{x}\sqrt{x}.$$

$\therefore$ $\sqrt{y}$ may be made to have the same surd factor as $\sqrt{x}$.

$\therefore$ $\sqrt{x}$ and $\sqrt{y}$ are *similar* surds.

But by hyp. they are dissimilar, which is absurd.

Therefore, etc.

41. *A surd cannot equal the sum or difference of a rational quantity and a surd.*

If possible let $\sqrt{a}=x\pm\sqrt{y}$;

$$\therefore\ a=x^2\pm 2x\sqrt{y}+y;$$

$$\therefore\ \sqrt{y}=\frac{a-x^2-y}{\pm 2x};$$

$\therefore$ a surd = a rational quantity, which is absurd.

Therefore, etc.

42. *A surd cannot equal the sum or difference of two dissimilar surds.*

If possible let $\sqrt{a}=\sqrt{x}\pm\sqrt{y}$, where $\sqrt{x}$ and $\sqrt{y}$ are dissimilar surds;

$$\therefore\ a=x\pm 2\sqrt{xy}+y;$$

$$\therefore\ \sqrt{xy}=\frac{a-x-y}{\pm 2};$$

$\therefore$ the product of two dissimilar surds is rational.

But this is impossible (40).

Therefore, etc.

43. **If a rational quantity and a surd are equal to a rational quantity and a surd, the rational parts are equal and the surd parts equal.**

For let $a+\sqrt{b}=x+\sqrt{y}$;

$$\therefore\ \sqrt{b}=(x-a)+\sqrt{y};$$

$\therefore$ unless $x-a=0$, we should have a surd equal to a rational quantity and a surd.

But this is impossible (41);

$$\left.\begin{array}{r}\therefore\ x-a=0;\ \ \therefore\ x=a\\ \text{whence }\sqrt{b}=\sqrt{y}\end{array}\right\}.$$

44. If $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then will

$$\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$$

For since $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$;

$$\therefore\ a+\sqrt{b}=x+2\sqrt{xy}+y;$$

$$\left.\begin{array}{r}\therefore\ a=x+y\\ \text{and}\ \sqrt{b}=2\sqrt{xy}\end{array}\right\}\ \text{(by 43)};$$

$$\therefore\ a-\sqrt{b}=x-2\sqrt{xy}+y;$$

$$\therefore\ \sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$$

45. The following is an application of the preceding theorem.

Find $\sqrt{4+2\sqrt{3}}$.

Let $\sqrt{4+2\sqrt{3}}=\sqrt{x}+\sqrt{y}$, ... (i).

$\therefore$ $\sqrt{4-2\sqrt{3}}=\sqrt{x}-\sqrt{y}$, ... (ii).

$\therefore$ squaring (i) and equating rational parts, we have

$$x+y=4, \quad \text{. . . . . (iii).}$$

Again, multiplying (i) and (ii) together, we have

$$x-y=\sqrt{16-12}=\sqrt{4}=2, \quad \text{. . . (iv),}$$

whence from (iii) and (iv) $x=3$, and $y=1$;

$$\therefore\ \sqrt{4+2\sqrt{3}}=\sqrt{3}+1.$$

46. If we had had to find $\sqrt{a-\sqrt{b}}$ we should have assumed this equal to $\sqrt{x}-\sqrt{y}$, and then have written down $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$.

From this it appears that $x-y$ is in either case equal to $\sqrt{a^2-b}$. *Hence the root cannot be found in the required form unless (a^2-b) is a perfect square.*

47. If $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}$, then will $\sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}$.

(Notice the different assumptions in 44 and 47.)

For since $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}$;

$$\therefore\ \text{cubing}\ a+\sqrt{b}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y};$$

$\therefore$ equating rational and surd parts,

$$a=x^3+3xy,$$

$$\text{and}\ \sqrt{b}=3x^2\sqrt{y}+y\sqrt{y};$$

$$\therefore\ a-\sqrt{b}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y};$$

$$\therefore\ \sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}.$$

48. To find $\sqrt[3]{a+\sqrt{b}}$ in the form $x+\sqrt{y}$.

$$\text{Since } \sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}, \quad . \quad . \quad \text{(i)},$$

$$\therefore\ \sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}, \quad . \quad . \quad \text{(ii)}.$$

Cubing (i) and equating rational parts, we have

$$x^3+3xy=a,$$

and multiplying (i) and (ii), we have

$$x^2-y=\sqrt[3]{a^2-b}=\sqrt[3]{p^3}, \text{ say}$$

$$=p;$$

$$\therefore\ y=x^2-p.$$

Hence $$x^3+3x(x^2-p)=a,$$

a cubic to be solved by trial. Now x being thus formed, and y being already known in terms of x, we have $x+\sqrt{y}$ the root required.

NOTE. a^2-b *in this case must be a perfect cube or the root cannot be found in the assumed form.*

Example. Find $\sqrt[3]{10+6\sqrt{3}}$.

$$\left.\begin{aligned}\text{Let } \sqrt[3]{10+6\sqrt{3}}&=x+\sqrt{y}\\ \therefore\ \sqrt[3]{10-6\sqrt{3}}&=x-\sqrt{y}\end{aligned}\right\} \quad . \quad . \quad \begin{aligned}&\text{(i)},\\&\text{(ii)},\end{aligned}$$

$$\text{Hence } x^3+3xy=10, \quad . \quad . \quad . \quad . \quad \text{(iii)},$$

$$\text{and } x^2-y=\sqrt[3]{100-108}=-2;$$

$$\therefore\ y=x^2+2;$$

$$\therefore \text{ from (iii) } x^3+3x(x^2+2)=10;$$

$$\therefore\ 4x^3+6x=10;$$

$$\therefore \text{ by trial } x=1,$$

$$\text{whence } y=3;$$

$$\therefore\ x+\sqrt{y}=1+\sqrt{3}.$$

EQUATIONS.

49. *A simple equation cannot have more than one root.*

For if possible let the general simple equation $ax+b=0$ have two roots, viz., α and β.

Then $a\alpha+b=0$, and $a\beta+b=0$;

$$\therefore\ a(\alpha-\beta)=0;$$

$$\therefore\ \text{either } a=0 \text{ or } \alpha-\beta=0.$$

Now a cannot $=0$ or b would also $=0$, and the expression $ax+b=0$ would not be an equation but an identity $0+0=0$;

$$\therefore\ \alpha-\beta=0;$$

$$\therefore\ \alpha=\beta.$$

Therefore there cannot be two *different* roots of a simple equation. In other words, a simple equation cannot have more than one root.

50. We shall presume on the student's knowledge of the ordinary methods of solving simple simultaneous equations of two or more unknowns. The following two sections will indicate methods of solution with which he may be unacquainted.

51. If $\left.\begin{array}{l} a_1x+b_1y+c_1z=0 \\ a_2x+b_2y+c_2z=0 \end{array}\right\}$ . . . . . . (i), and . . . . . . (ii),

then will $\dfrac{x}{b_1c_2-b_2c_1}=\dfrac{y}{c_1a_2-c_2a_1}=\dfrac{z}{a_1b_2-a_2b_1}$.

For multiplying (i) by c_2 and (ii) by c_1 and subtracting, we obtain $(c_2a_1-c_1a_2)x+(b_1c_2-b_2c_1)y=0$;

$$\therefore\ (b_1c_2-b_2c_1)y=(c_1a_2-c_2a_1)x,$$

whence $\dfrac{x}{b_1c_2-b_2c_1}=\dfrac{y}{c_1a_2-c_2a_1}$. Similarly either of these may be proved, equal to the third fraction.

Example. Solve
$$\left.\begin{aligned} ax+by+cz&=0 \\ x+y+z&=0 \\ \frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}&=1 \end{aligned}\right\} \quad \begin{aligned} &\ldots\ldots (i), \\ &\ldots\ldots (ii), \\ &\ldots\ldots (iii). \end{aligned}$$

From (i) and (ii)

$$\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}.$$

$\therefore$ multiplying (iii) by either of these fractions inverted, we have

$$1+1+1=\frac{b-c}{x};$$

$$\therefore x=\frac{b-c}{3}, \qquad \text{so } y=\frac{c-a}{3}, \qquad \text{and } z=\frac{a-b}{3}.$$

52. To solve simultaneous equations of three unknowns by means of arbitrary multipliers.

Let the equations be

$$a_1x+b_1y+c_1z=d_1, \quad \ldots \quad (i),$$
$$a_2x+b_2y+c_2z=d_2, \quad \ldots \quad (ii),$$
$$a_3x+b_3y+c_3z=d_3, \quad \ldots \quad (iii).$$

Multiply (i) by l, (ii) by m, (iii) by n, and add;

$$\therefore (la_1+ma_2+na_3)x+(lb_1+mb_2+nb_3)y+(lc_1+mc_2+nc_3)z$$
$$=ld_1+md_2+nd_3.$$

Now *assume* that $lb_1+mb_2+nb_3=0,$

and $lc_1+mc_2+nc_3=0;$

$$\therefore \frac{l}{b_2c_3-b_3c_2}=\frac{m}{b_3c_1-b_1c_3}=\frac{n}{b_1c_2-b_2c_1}, \quad . \quad (51);$$

$\therefore$ if we multiply the equations in order by $b_2c_3-b_3c_2$, $b_3c_1-b_1c_3$, and $b_1c_2-b_2c_1$, and add, our assumptions will be justified and the terms involving y and z will vanish.

Example. Find x from

$$\begin{aligned} x+2y+3z&=14, & b_2c_3-b_3c_2&=\ (3.1)-(-1.4)=\ 7,\\ 2x+3y+4z&=20, & b_3c_1-b_1c_3&=(-1.3)-\ (2.1)=-5,\\ x-y+z&=\ 2, & b_1c_2-b_2c_1&=\ (2.4)-\ (3.3)=-1. \end{aligned}$$

Multiply in order by these and we have

$$\begin{aligned} 7x+14y+21z&=\ 98,\\ -10x-15y-20z&=-100,\\ -x+y-z&=-\ 2;\\ \therefore\ -4x=-4;&\qquad \therefore\ x=1. \end{aligned}$$

53. *A quadratic cannot have more than two roots.*

For let the general quadratic be $ax^2+bx+c=0$, and if possible let it have three roots α, β, γ;

$$\begin{aligned} \therefore\ a\alpha^2+b\alpha+c&=0,\\ a\beta^2+b\beta+c&=0,\\ a\gamma^2+b\gamma+c&=0.\\ \therefore\ a(\alpha^2-\beta^2)+b(\alpha-\beta)&=0;\\ \therefore\quad a(\alpha+\beta)+b&=0.\\ \text{So}\quad a(\alpha+\gamma)+b&=0;\\ \therefore\ a(\beta-\gamma)&=0. \end{aligned}$$

$\therefore$ either $a=0$ or $(\beta-\gamma)=0$, but a cannot $=0$ or the quadratic would degenerate into a simple equation;

$$\therefore\ \beta-\gamma=0;\quad \therefore\ \beta=\gamma.$$

$\therefore$ There are not three different roots.

Therefore, etc.

54. To solve the general quadratic,

$$\begin{aligned} ax^2+bx+c&=0,\\ ax^2+bx&=-c.\\ \therefore\ x^2+\frac{b}{a}x&=-\frac{c}{a};\\ \therefore\ x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2&=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}; \end{aligned}$$

$$\therefore\ x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a};$$

$$\therefore\ x=\frac{\pm\sqrt{b^2-4ac}-b}{2a}.$$

This method reduces the equation to the *unity form*, i.e. x^2 is made to have 1 for its coefficient, and then the square of half the coefficient of x is added to both sides, so making the left-hand side a perfect square.

By remembering the above formula, we write down the value of x at once.

Thus $2x^2+3x-14=0$;

$$\therefore\ x=\frac{\pm\sqrt{9+112}-3}{4}=\frac{\pm 11-3}{4},$$

$$=2 \text{ or } -3\tfrac{1}{2}.$$

55. Consider the equation $(x-a)(x-b)=0$.

It is clear that if *either* factor$=0$ the equation is satisfied.

But if $x-a=0$, $\quad x=a$,

and if $x-b=0$, $\quad x=b$;

$\therefore$ a and b are the two roots of the equation.

Hence it appears that when an equation can be factorized, the quantity on the right-hand side being 0, we shall obtain the roots required simply by putting each factor $=0$; and conversely if $x=a$ or $x=b$, the equation is $(x-a)(x-b)=0$.

The same is obviously true for any number of roots.

Thus the equation which has for its roots 0, 1, and -1 is

$$x(x-1)(x+1)=0.$$

56. The method of factorization is extremely useful when the coefficient of x^2 is large,

e.g. $\qquad 110x^2-21x+1=0$;

$$\therefore\ (11x-1)(10x-1)=0;$$

$$\therefore\ x=\frac{1}{11} \text{ or } \frac{1}{10}.$$

If we had reduced the equation to the unity form we should

have $x^2-\frac{21x}{110}=-\frac{1}{110}$, and then to each side we should have had to add $\left(\frac{21}{220}\right)^2$.

When however the coefficient of x^2 is a square, some labour may be saved by proceeding as in the extraction of square root direct. Thus if $9x^2-11x-364=0$.

Extract the square root of $9x^2-11x$,

$$
\begin{array}{r|l}
 & 9x^2-11x\,(\,3x-\frac{11}{6} \\
 & 9x^2 \\
\hline
6x-\frac{11}{6} & -11x \\
 & -11x+\left(\frac{11}{6}\right)^2. \\
\hline
\end{array}
$$

$\therefore$ the term $+\left(\frac{11}{6}\right)^2$ is wanting, so that we have

$$9x^2-11x+\left(\frac{11}{6}\right)^2=364+\left(\frac{11}{6}\right)^2=\frac{13225}{36};$$

$$\therefore\ 3x-\frac{11}{6}=\pm\frac{115}{6};$$

$$\therefore\ 3x=21 \text{ or } -\frac{104}{6};$$

$$\therefore\ x=7 \text{ or } -\frac{52}{9}.$$

57. **Whenever we can divide both sides of an equation by any quantity involving the unknown, we shall obtain a root or roots of the equation by putting this quantity equal to zero.**

For let the left-hand side of the equation be $(x-a)P$, and the right $(x-a)Q$;

$$\therefore\ (x-a)P=(x-a)Q;$$

$$\therefore\ (x-a)(P-Q)=0;$$

$$\therefore\ x-a=0 \text{ will give one root (55).}$$

If the factor common to both sides of the equation had been a quadratic, we should have obtained *two* values for the unknown, and generally the *degree of the divisor will indicate the number of roots obtained by equating it to zero.*

58. We have thus pointed out three methods of solving a quadratic :—

(i) The unity form method.
(ii) Factorizing.
(iii) Formula.

59. **To find** $(x-a_1)(x-a_2)(x-a_3)\ .\ .\ .\ (x-a_n)$.

By actual multiplication

$$(x-a_1)(x-a_2)=x^2-(a_1+a_2)x+a_1a_2$$

$$(x-a_1)(x-a_2)(x-a_3)=x^3-(a_1+a_2+a_3)x^2 +(a_1a_2+a_2a_3+a_3a_1)x-a_1a_2a_3.$$

Now, here four laws appear to hold :—

(i) The highest index of x denoted by the number of factors.
(ii) The indices of x decrease regularly by unity.
(iii) The coefficients are 1, the sum of a_1, a_2 . . . singly, the sum of their products, two together, the sum of their products, three together, and so on.
(iv) The signs are alternate.

Now *assume* these laws to hold for $\overline{n-1|}$ factors. That is, let

$$(x-a_1)(x-a_2)\ldots(x-a_{n-1})=x^{n-1}-(a_1+a_2+\ldots+a_{n-1})x^{n-2} +(a_1a_2+\ldots)x^{n-3}-\ldots+(-1)^{n-1}(a_1a_2\ldots a_{n-1}).$$

Multiply both sides by $(x-a_n)$, and we have

$$(x-a_1)(x-a_2)\ldots(x-a_n)=x^n-(a_1+a_2+\ldots+a_n)x^{n-1} +(a_1a_2+\ldots)x^{n-2}-\ldots+(-1)^n(a_1a_2\ldots a_n).$$

$\therefore$ If our assumed laws hold good for $(n-1)$ factors, so also do they hold good for n factors; but they *are* true for three factors; therefore for four factors; therefore for five factors; and so by mathematical induction universally.

60. Connexion between the coefficients of an equation and its roots.

Let $a_1, a_2, a_3, \ldots a_n$ be the roots.

$\therefore$ the equation is $(x-a_1)(x-a_2)\ldots(x-a_n)=0$,

or $x^n-(a_1+a_2+\ldots+a_n)x^{n-1}+(a_1a_2+a_2a_3+\ldots)x^{n-2}-\ldots$
$+(-1)^n(a_1a_2\ldots a_n)=0$.

$\therefore$ The sum of the roots = coefficient of second highest power of x with sign changed.

The sum of their products, two together, = coefficient of next highest power of x without sign changed.

The sum of their products, three together, = coefficient of next highest power of x with sign changed;

and so on alternately in sign.

Note.—The equation is in the unity form for this connexion to hold.

61. We may easily prove the preceding theorem true in the special case of the quadratic $ax^2+bx+c=0$.

If α and β are the roots of this, we have

$$\alpha=\frac{\sqrt{b^2-4ac}-b}{2a},$$

$$\beta=\frac{-\sqrt{b^2-4ac}-b}{2a},$$

$$\therefore\ \alpha+\beta=-\frac{2b}{2a}=-\frac{b}{a},\ \text{and}\ \alpha\beta=\frac{4ac}{4a^2}=\frac{c}{a}.$$

The same results which obtain from the Theorem in 60.

Problems on 60 are frequent. We append one.

If α, β, γ are the roots of $x^3-px^2+qx-r=0$.

Prove $\alpha+\frac{1}{\alpha}+\beta+\frac{1}{\beta}+\gamma+\frac{1}{\gamma}=\frac{pr+q}{r}$.

For since $\alpha+\beta+\gamma=p$, ... (i),

$\alpha\beta+\beta\gamma+\gamma\alpha=q$, ... (ii),

$\alpha\beta\gamma=r$, ... (iii);

$\therefore$ from (ii) and (iii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=\dfrac{q}{r}$;

$$\therefore\ \alpha+\frac{1}{\alpha}+\beta+\frac{1}{\beta}+\gamma+\frac{1}{\gamma}=p+\frac{q}{r}=\frac{pr+q}{r}.$$

62. Since the roots of the quadratic $ax^2+bx+c=0$ are

$$\frac{\pm\sqrt{b^2-4ac}-b}{2a},$$

it is clear that the roots will be real and different, real and equal, or impossible, according as b^2 is greater than, equal to, or less than $4ac$.

63. The following equations, in each of which there is some particular artifice, should be carefully observed.

Homogeneous Equations.

(i) $x^2+y^2=13, \qquad x^2+xy=10.$

Let $y=vx$;

$\therefore\ x^2(1+v^2)=13, \qquad$ and $x^2(1+v)=10$;

$$\therefore\ \frac{1+v^2}{1+v}=\frac{13}{10};$$

$$\therefore\ 10+10v^2=13+13v;$$

$$\therefore\ 10v^2-13v-3=0;$$

$$\therefore\ (5v+1)(2v-3)=0;$$

$$\therefore\ v=-\frac{1}{5}\ \text{or}\ \frac{3}{2}.$$

Hence

$$x^2\left(1-\frac{1}{5}\right)=10;\quad \therefore\ x^2=\frac{25.2}{4}=\pm\frac{5}{2}\sqrt{2};\quad \therefore\ y=\mp\frac{1}{2}\sqrt{2};$$

or

$$x^2\left(1+\frac{3}{2}\right)=10;\quad \therefore\ x^2=4;\quad \therefore\ x=\pm2;\quad \therefore\ y=\pm3.$$

This method is useful when the equations are Homogeneous.

Adding and Subtracting.

$$\text{(ii)}\quad \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}=2;$$

$\therefore$ adding and subtracting, we have

$$\frac{2\sqrt{x^2+1}}{2\sqrt{x^2-1}}=\frac{3}{1};$$

$$\therefore\ \frac{x^2+1}{x^2-1}=\frac{9}{1}.$$

$\therefore$ adding and subtracting again,

$$\frac{2x^2}{2}=\frac{10}{8};\qquad \therefore\ x^2=\frac{5}{4};\qquad \therefore\ x=\pm\frac{1}{2}\sqrt{5}.$$

Solving in Reciprocals.

$$\text{(ii)}\quad a(x+y)=xy, \quad \ldots\ldots \quad \text{(i)},$$
$$b(y+z)=yz, \quad \ldots\ldots \quad \text{(ii)},$$
$$c(z+x)=zx, \quad \ldots\ldots \quad \text{(iii)}.$$

$$\therefore\ \frac{1}{x}+\frac{1}{y}=\frac{1}{a},\quad \text{dividing (i) by } axy,$$
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{b},\quad \text{,, (ii) by } byz,$$
$$\frac{1}{z}+\frac{1}{x}=\frac{1}{c},\quad \text{,, (iii) by } czx.$$

$\therefore$ adding the first and third and subtracting the second of these

$$\frac{2}{x}=\frac{1}{a}+\frac{1}{c}-\frac{1}{b}=\frac{bc+ab-ac}{abc};$$

$$\therefore\ x=\frac{2abc}{bc-ca+ab}.$$

So by *symmetry* $$y=\frac{2abc}{bc+ca-ab},$$

and $$z=\frac{2abc}{ca+ab-bc}.$$

A perfect square of $x+\frac{1}{x}$ by adding two.

(iv) $x^4+1=0$;

$$\therefore\ x^2+\frac{1}{x^2}=0, \text{ dividing by } x^2;$$

$$\therefore\ x^2+2+\frac{1}{x^2}=2;$$

$$\therefore\ x+\frac{1}{x}=\pm\sqrt{2};$$

$$\therefore\ x^2\pm\sqrt{2}.\ x+1=0,$$

which gives two quadratics which may be easily solved.

Quadratic in an expression.

(v) $4x^2+8x+\frac{8}{x}+\frac{4}{x^2}=37$;

$$\therefore\ 4\left(x^2+\frac{1}{x^2}\right)+8\left(x+\frac{1}{x}\right)=37.$$

Add 8 to both sides, and we have

$$4\left(x+\frac{1}{x}\right)^2+8\left(x+\frac{1}{x}\right)=45.$$

Let $x+\frac{1}{x}=y$;

$$\therefore\ 4y^2+8y-45=0;$$

$$\therefore\ (2y-5)(2y+9)=0;$$

$$\therefore\ y=\frac{5}{2}, \text{ or } -\frac{9}{2}.$$

(i) Let $x+\frac{1}{x}=\frac{5}{2}$; $\quad\therefore\ 2x^2-5x+2=0$;

$$\therefore\ (x-2)(2x-1)=0;\ \therefore\ x=2 \text{ or } \tfrac{1}{2}.$$

(ii) Let $x+\frac{1}{x}=-\frac{9}{2}$; $\quad\therefore\ 2x^2+9x+2=0$;

$$\therefore\ x=\frac{\pm\sqrt{81-16}-9}{4}$$

$$=\frac{\pm\sqrt{65}-9}{4}.$$

Two heavy roots.

(vi) $\sqrt{x^2+4x+13}+\sqrt{x^2+4x-8}=7.$

Now $(x^2+4x+13)-(x^2+4x-8)=21$, evidently;

$\therefore$ dividing each side of this expression by the corresponding sides of the given equation,

$$\sqrt{x^2+4x+13}-\sqrt{x^2+4x-8}=3;$$

$\therefore$ adding this to the given equation

$2\sqrt{x^2+4x+13}=10$; $\therefore$ $\sqrt{x^2+4x+13}=5.$

Therefore $x^2+4x-12=0.$

Therefore $(x-2)(x+6)=0.$

Therefore $x=2$ or $-6.$

Factorizing.

(vii) $x^3-3x=a^3+\dfrac{1}{a^3},$

$$=\left(a+\frac{1}{a}\right)^3-3\left(a+\frac{1}{a}\right).$$

Let $a+\dfrac{1}{a}=b,$

Whence $x^3-3x=b^3-3b.$

Therefore $x^3-b^3=3(x-b)$; $\therefore$ $x-b=0.$

$\therefore$ $x=b=a+\dfrac{1}{a},$

Therefore $x^2+bx+b^2=3$, a quadratic in x, whence the two other roots may be found at once.

Arbitrary Multipliers.

(viii)
$$\begin{aligned}
x+y+z+ \ldots\ldots &=1,\\
ax+by+cz+ \ldots &=d,\\
a^2x+b^2y+c^2z+ \ldots &=d^2,\\
\ldots\ldots &= \ldots\\
a^nx+b^ny+c^nz+ \ldots &=d^n.
\end{aligned}$$

Multiplying these in order, beginning from the bottom, by $1, m_1, m_2, \ldots m_n$ and adding, we obtain

$$x(a^n+m_1a^{n-1}+\ldots+m_n)+y(b^n+m_1b^{n-1}+\ldots+m_n)+\ldots$$
$$=(d^n+m_1d^{n-1}+\ldots+m_n).$$

Equate to zero the coefficients of all the unknowns except x, and we obtain

$$x=\frac{d^n+m_1d^{n-1}+\ldots+m_n}{a^n+m_1a^{n-1}+\ldots+m_n}.$$

Now consider the equation

$$t^n+m_1t^{n-1}+\ldots+m_n=0.$$

We *know* that this is satisfied by each of the n quantities $b, c, \ldots$ for this we have assumed;

$$\therefore\ t^n+m_1t^{n-1}+\ldots+m_n=(t-b)(t-c)\ldots \text{ to } n \text{ factors},$$
$$\therefore\ d^n+m_1d^{n-1}+\ldots+m_n=(d-b)(d-c)\ldots \quad ,,$$
and $a^n+m_1a^{n-1}+\ldots+m_n=(a-b)(a-c)\ldots$,,

$$\therefore\ x=\frac{(d-b)(d-c)\ldots}{(a-b)(a-c)\ldots}.$$

(ix) $\left(\dfrac{x-a}{x-b}\right)^3=\dfrac{x-2a+b}{x-2b+a}$;

$$\therefore\ \left(\frac{x-a}{x-b}\right)^2=\frac{x^2-2ax+bx-bx+2ab-b^2}{x^2-2bx+ax-ax+2ab-a^2},$$
$$=\frac{(x-a)^2-(a-b)^2}{(x-b)^2-(a-b)^2};$$
$$\therefore\ 1=\frac{1-\left(\dfrac{a-b}{x-a}\right)^2}{1-\left(\dfrac{a-b}{x-b}\right)^2};$$
$$\therefore\ \frac{a-b}{x-a}=\pm\frac{a-b}{x-b},$$
$$x-a=-x+b,$$
$$2x=a+b,$$
$$x=\frac{a+b}{2}.$$

Note.—If we take the positive sign above we obtain no further value for x, as the equation then becomes $a=b$.

(x) Symmetrical equations may often be solved by putting $x=p+q$ and $y=p-q$.

Sometimes it is convenient to find xy, yz, and zx; to multiply any two of these and divide by the third, so obtaining x^2, y^2, and z^2.

Sometimes it is advantageous to reduce a fraction into the *mixed form*. Thus, if we have for our equation

$$\frac{132x+1}{3x+1}+\frac{8x+5}{x-1}=52,$$

we may write

$$44-\frac{43}{3x+1}+8+\frac{13}{x-1}=52,$$

$$\text{whence}\quad \frac{13}{x-1}=\frac{43}{3x+1},$$

and the solution is very easily effected.

Note.—In distance problems remember,

$$\text{rate}\times\text{time}=\text{distance},$$

$$\text{whence rate}=\frac{\text{distance}}{\text{time}},$$

$$\text{and time}=\frac{\text{distance}}{\text{rate}},$$

and always be careful to specify the *units* of time and space.

PROGRESSIONS.

64. Quantities are said to be in arithmetical progression when they proceed by a common difference.

Thus $1+4+7+\ldots$ are in A.P. with a common difference 3.

The first term is denoted by a, the last by l, the number of terms by n, their sum by S, and their common difference by d.

Observe. To find d write (second term)—(first term),
or (third term)—(second term),
and so on.

Thus in the series $-5-3-1+\ldots$ $d=-3-(-5)=2$.

65. $l=a+\overline{n-1}\,|\,d.$

For since the series is $a+(a+d)+(a+2d)+(a+3d)+\ldots$ it is plain that each term is $a+(\quad)d$, and the blank bracket is all we have to fill up.

Now the coefficient of d in any term is obviously *one less than the number of that term.*

$\therefore$ in the nth term, which is the last, it will be $(n-1)$;

$$\therefore\ l=a+(n-1)d.$$

66. $$S=(a+l)\frac{n}{2}=\{2a+\overline{n-1}\,|\,d\}\frac{n}{2}.$$

For $S=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l,$

and $S=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a;$

$$\therefore\ 2S=(a+l)+(a+l)+\ \ldots\ \text{to } n \text{ terms},$$
$$=(a+l)n;$$
$$\therefore\ S=(a+l)\frac{n}{2};$$
$$=\{a+a+(n-1)d\}\frac{n}{2}=\{2a+(n-1)d\}\frac{n}{2}.$$

67. **To find** $a^2+(a+d)^2+(a+2d)^2+\ldots+\{a+(n-1)d\}^2.$

Let $a=A_1,\ a+d=A_2\ \ldots\ a+(n-1)d=A_n,$

and let $S=A_1^2+A_2^2+A_3^2+\ \ldots\ +A_n^2.$

Now $A_2^3-A_1^3=(A_1+d)^3-A_1^3=3A_1^2d+3A_1d^2+d^3,$

$A_3^3-A_2^3=(A_2+d)^3-A_2^3=3A_2^2d+3A_2d^2+d^3,$

$A^3_{n+1}-A_n^3=(A_n+d)^3-A_n^3=3A^2_nd+3A_nd^2+d^3.$

$\therefore$ adding, we obtain

$$A^3_{n+1}-A^3_1=3dS+3d^2(A_1+A_2+\ \ldots\ +A_n)+nd^3;$$

$$\therefore\ (a+nd)^3-a^3=3dS+3d^2\{2a+(n-1)d\}\frac{n}{2}+nd^3,$$

whence on reduction,

$$S=na\{a+(n-1)d\}+\frac{nd^2}{6}(n-1)(2n-1)$$

Cor. $1^2+2^2+3^2+ \ldots +n^2=\frac{n.(n+1)(2n+1)}{6}$.

For writing $a=1$ and $d=1$ in the former result, we have

$$n\{1+(n-1)\}+\frac{n}{6}(n-1)(2n-1),$$

$$=n^2+\frac{n}{6}(n-1)\}(2n-1)=\frac{n}{6}(6n+2n^2-3n+1),$$

$$=\frac{n}{6}(2n^2+3n+1)=\frac{1}{6}n(n+1)(2n+1).$$

68. To find the sum to n terms of a series the rth term of which, $a_ra_{r+1}a_{r+2} \ldots a_{r+s-1}$ consists of s factors in A.P.

Let d be the common difference of the factors;

$$\therefore\ a_{r+s}-a_{r-1}=(s+1)d;$$

$$\therefore\ \frac{a_{r+s}-a_{r-1}}{(s+1)d}=1.$$

Now $a_1a_2\ldots a_s=a_1a_2\ldots a_s\times\frac{a_{s+1}-a_0}{(s+1)d}$

$$=\frac{1}{(s+1)d}\{a_1a_2\ldots a_{s+1}-a_0a_1\ldots a_s\},$$

$$a_2a_3\ldots a_{s+1}=a_2a_3\ldots a_{s+1}\times\frac{a_{s+2}-a_1}{(s+1)d}$$

$$=\frac{1}{(s+1)d}\{a_2a_3\ldots a_{s+2}-a_1\ldots a_{s+1}\},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad=\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$a_na_{n+1}\ldots a_{n+s-1}=a_na_{n+1}\ldots a_{n+s-1}\times\frac{a_{n+s}-a_{n-1}}{(s+1)d}$$

$$=\frac{1}{(s+1)d}\{a_n\ldots a_{n+s}-a_{n-1}\ldots a_{n+s-1}\};$$

$$\therefore\ S=\frac{1}{(s+1)d}\{a_n\ldots a_{n+s}-a_0\ldots a_s\},$$

Hence the rule.—For N^r take the last term with one factor

added to the end—the first term with one factor added to the beginning.

For D^r take one more than the number of factors in each term multiplied by the common difference.

69. *Ex.* (i) $1.4.7+4.7.10+$ etc., to n terms,

$$=\frac{1}{4.3}\{(3n-2)(3n+1)(3n+4)(3n+7)-(-2).1.4.7\},$$

$$=\text{etc.}$$

Ex. (ii). Find $1+3+6+10+15+21+\ \ldots$ to n terms.

$$\text{This series }=\tfrac{1}{2}\{2+6+12+20+30+42+\ \ldots\ \},$$
$$=\tfrac{1}{2}\{1.2+2.3+3.4+4.5+\ \ldots\ +n(n+1)\},$$
$$=\tfrac{1}{2}\cdot\frac{1}{3.1}\{n.(n+1)(n+2)-0.1.2\},$$
$$=\frac{n.(n+1)(n+2)}{6}.$$

Ex. (iii). Find

$$1.2.n+2.3(n-1)+3.4(n-2)+\ \ldots\ +n(n+1).1.$$

Let $f(n)=1.2.n+2.3(n-1)+3.4(n-2)+\ \ldots$ to n terms;

$$\therefore\ f(n-1)=1.2.(n-1)+2.3(n-2)+3.4(n-3)+\ \ldots\ \text{to }(n-1)\text{ terms};$$

$$\therefore\ f(n)-f(n-1)=1.2+2.3+\ \ldots\ +n(n+1),$$
$$=\tfrac{1}{3}.n(n+1)(n+2).$$

Put $n=1, 2, 3, \ldots$ successively, and we have

$$f(1)-f(0)=\tfrac{1}{3}.1.2.3,$$
$$f(2)-f(1)=\tfrac{1}{3}.2.3.4,$$
$$f(3)-f(2)=\tfrac{1}{3}.3.4.5,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$f(n)-f(n-1)=\tfrac{1}{3}n.(n+1)(n+2),$$

whence adding *and remembering that* $f(0)=0$, we have

$$f(n)=\tfrac{1}{3}.\tfrac{1}{4}.n(n+1)(n+2)(n+3)$$
$$=\frac{n(n+1)(n+2)(n+3)}{12}.$$

70. To find the sum to n terms of a series, the rth term of which $\dfrac{1}{a_r a_{r+1}\ \ldots\ a_{r+s-1}}$ has a denominator of s factors, themselves in A.P.

Let d be the common difference of the factors.

Then $a_{r+s-1}-a_r=(s-1)d$;

$$\therefore\ \frac{a_{r+s-1}-a_r}{(s-1)d}=1.$$

Now
$$\frac{1}{a_1\ldots a_s}=\frac{1}{a_1\ldots a_s}\times\frac{a_s-a_1}{(s-1)d}$$
$$=\frac{1}{(s-1)d}\left\{\frac{1}{a_1\ldots a_{s-1}}-\frac{1}{a_2\ldots a_s}\right\},$$

$$\frac{1}{a_2\ldots a_{s+1}}=\frac{1}{a_2\ldots a_{s+1}}\times\frac{a_{s+1}-a_1}{(s-1)d}$$
$$=\frac{1}{(s-1)d}\left\{\frac{1}{a_2\ldots a_s}-\frac{1}{a_3\ldots a_{s+1}}\right\},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad=\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\frac{1}{a_n\ldots a_{n+s-1}}=\frac{1}{a_n\ldots a_{n+s-1}}\times\frac{a_{n+s-1}-a_n}{(s-1)d}$$
$$=\frac{1}{(s-1)d}\left\{\frac{1}{a_n\ldots a_{n+s-2}}-\frac{1}{a_{n+1}\ldots a_{n+s-1}}\right\};$$

$\therefore$ adding we have

$$S=\frac{1}{(s-1)d}\left\{\frac{1}{a_1\ldots a_{s-1}}-\frac{1}{a_{n+1}\ldots a_{n+s-1}}\right\}.$$

Hence the rule.—Take away the last factor from the first term and the first factor from the last term, and divide the difference by one less than the number of factors in each term multiplied by the common difference.

Example.

$$\frac{1}{1.3.5}+\frac{1}{3.5.7}+\text{etc., to } n \text{ terms}=\frac{1}{2.2}\left\{\frac{1}{1.3}-\frac{1}{(2n+1)(2n+3)}\right\}.$$

71. It will be observed in sections 68 and 70 that the first factor of any term is the second factor of the preceding one. By the simple artifice, however, of splitting up the nth term as in the following example into two others, such that the above condition is satisfied, we can apply the formula in some other cases.

(i) Find $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+$ etc.

$$\text{The } n\text{th term} = \frac{1}{n(n+3)} = \frac{(n+1)(n+2)}{n(n+1)(n+2)(n+3)},$$

$$= \frac{(n^2+3n)+2}{n(n+1)(n+2)(n+3)},$$

$$= \frac{1}{(n+1)(n+2)} + \frac{2}{n(n+1)(n+2)(n+3)}.$$

$$\therefore \; S = \frac{1}{1.1}\left\{\frac{1}{2} - \frac{1}{n+2}\right\} + \frac{2}{3.1}\left\{\frac{1}{1.2.3} - \frac{1}{(n+1)(n+2)(n+3)}\right\}$$

$=$ etc.

Cor. Hence the sum *ad inf.*, $=\frac{1}{2}+\frac{1}{9}=\frac{11}{18}$.

72. Quantities are said to be in *Geometric Progression* when they proceed by a common ratio.

Thus $1+3+9+\ \ldots$ are in G.P., and the common ratio $r=3$.

The typical series is $a+ar+ar^2+ar^3+\ \ldots$

73. $l=ar^{n-1}$.

For the index of r in any term being one less than the number of that term, it is plain that the nth term is ar^{n-1}; $\therefore\ l=ar^{n-1}$, for l is the nth term.

74. $\quad S=\frac{a(r^n-1)}{r-1}=\frac{rl-a}{r-1}.$

For $S=a+ar+ar^2+\ \ldots\ +ar^{n-1}$;

$\therefore\ rS=\quad ar+ar^2+\ \ldots\ +ar^{n-1}+ar^n$;

$\therefore\ S(r-1)=a(r^n-1)$; $\therefore\ S=\dfrac{a(r^n-1)}{r-1}=\dfrac{ar^n-a}{r-1}=\dfrac{rl-a}{r-1}$.

Note.—If r be <1 it will be convenient to employ for S the forms $\dfrac{a(1-r^n)}{1-r}=\dfrac{a-rl}{1-r}$.

75. The sum of a Geometric series, *ad inf.* **Notion of a Limit.**

(i) If r be >1 or if $r=1$, the sum is evidently infinite.

(ii) If r be <1.

Then there is some quantity, *called the Limit*, which the series approaches nearer and nearer as we increase the number of terms taken, which it never *quite* reaches, but from which it may be made to differ *as little as we please* by sufficiently increasing the number of terms.

Thus the limit of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \ldots$ *ad inf.* $=2$.

Now it is plain that the more terms we take the nearer it approaches to 2.

For the first term falls short of 2 by 1,

the sum of the first two terms " $\dfrac{1}{2}$,

" three " $\dfrac{1}{2^2}$,

" n " $\dfrac{1}{2^{n-1}}$.

Therefore the sum never reaches 2, and yet since we may make $\dfrac{1}{2^{n-1}}$ as small as we please by sufficiently increasing n, it may be made to differ from 2 as little as we please;

$\therefore$ 2 is the *limit* of the series.

Note.—A proper fraction raised to an infinite power has 0 for its limit.

76. To find $a+ar+ar^2+\ldots$ *ad inf.*, r being a proper fraction.

Denote the sum by Σ;

$\therefore\ \Sigma=$ the limit of $\dfrac{a(1-r^n)}{1-r}$ when n is made infinite,

$\therefore$ but then r^n has 0 for its limit;

$$\therefore\ \Sigma=\frac{a}{1-r}.$$

Recurring decimals are in reality G.S. *ad inf.*

For $.\dot{2}=\dfrac{2}{10}+\dfrac{2}{100}+\ldots$ *ad inf.*

$$=\frac{\frac{2}{10}}{1-\frac{1}{10}}=\frac{2}{9}.$$

77. **Def.** Quantities $a, b, c, d, \ldots$ are said to be in Harmonic progression when $a:c::a-b:b-c$, and so on.

78. The reciprocals of quantities in H.P. are in A.P.

For let abc be in H.P.;

$$\therefore\ a:c::a-b:b-c;$$

$$\therefore\ ab-ac=ac-bc;$$

$$\therefore\ \frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}\text{, dividing by } abc;$$

$$\therefore\ \frac{1}{a},\ \frac{1}{b},\ \frac{1}{c}\text{, are in A.P.}$$

This leads to the following practical rule. In dealing with quantities in H.P., convert them into their reciprocals, work in A.P., and then reconvert.

Observe.—This cannot however be applied to find the sum of any number of quantities in H.P., for their sum is not the reciprocal of the sum of their reciprocals, *e.g.* $2+4$ is not $\dfrac{1}{\frac{1}{2}+\frac{1}{4}}$.

79. *The* A.G. *and* H. *means between a and b.*

(i) Let x=the A.M.; $\therefore x-a=b-x$; $\therefore 2x=a+b$;

$$\therefore x=\frac{a+b}{2}.$$

(ii) Let x=the G.M.; $\therefore \frac{x}{a}=\frac{b}{x}$; $\therefore x^2=ab$;

$$\therefore x=\sqrt{ab}.$$

(iii) Let x=the H.M.; $\therefore \frac{1}{a}, \frac{1}{x}, \frac{1}{b}$, are in A.P.

$$\therefore \frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x}; \qquad \therefore \frac{2}{x}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab},$$

$$\therefore x=\frac{2ab}{a+b}.$$

80. To insert n A.M. between a and b.

Since here $b=a+(n+1)d$; $\therefore d=\frac{b-a}{n+1}$;

$\therefore$ the n means are

$$a+\frac{b-a}{n+1}, \quad a+\frac{2b-2a}{n+1} \ldots a+\frac{nb-na}{n+1},$$

which become on simplifying

$$\frac{na+b}{n+1}, \quad \frac{(n-1)a+2b}{n+1} \ldots \frac{a+nb}{n+1}.$$

The coefficients of a in the numerators regularly *decreasing* by 1 and the coefficients of b similarly *increasing.*

81. To insert n H.M. between a and b.

Since here

$$\frac{1}{b}=\frac{1}{a}+(n+1)d; \quad \therefore \frac{1}{b}-\frac{1}{a}=\overline{n+1}\,|\,d; \quad \therefore d=\frac{a-b}{ab(n+1)};$$

$\therefore$ the reciprocals of the required means are

$$\frac{1}{a}+\frac{a-b}{ab(n+1)}, \quad \frac{1}{a}+\frac{2a-2b}{ab(n+1)} \ldots \frac{1}{a}+\frac{na-nb}{ab(n+1)}.$$

These become on simplifying,

$$\frac{nb+a}{ab(n+1)}, \quad \frac{(n-1)b+2a}{ab(n+1)} \quad \ldots \quad \frac{b+na}{ab(n+1)};$$

$\therefore$ the H.M. required are

$$\frac{ab(n+1)}{nb+a}, \quad \frac{ab(n+1)}{(n-1)b+2a} \quad \ldots \quad \frac{ab(n+1)}{b+na}.$$

The coefficients of b in the D^r regularly decreasing by 1, and the coefficients of a similarly increasing.

82. The following are a few examples in series:—

(i) n A.M., n G.M., and n H.M. are inserted between a and b; show that the product of the n G.M. is a geometrical mean between the products of the n A.M. and the n H.M.

Let $a_1 a_2 \ldots a_n$ be the n A.M.,
$h_1 h_2 \ldots h_n$ " n H.M.,
$g_1 g_2 \ldots g_n$ " n G.M.

Now $g_1 g_2 \ldots g_n = ar.ar^2 \ldots ar^n = a^n \cdot r^{1+2+\ldots+n}$,

$$= a^n r^{\frac{n(n+1)}{2}}.$$

But since $b = ar^{n+1}$; $\quad \therefore r^{n+1} = \frac{b}{a}$; $\quad \therefore r^{\frac{n(n+1)}{2}} = \left(\frac{b}{a}\right)^{\frac{n}{2}}$;

$$\therefore g_1 g_2 \ldots g_n = a^n \left(\frac{b}{a}\right)^{\frac{n}{2}} = (ab)^{\frac{n}{2}} = \sqrt{a^n b^n}, \quad . \qquad \text{(i).}$$

$$\text{Now } a_1 a_2 \ldots a_n = \frac{na+b}{n+1} \cdot \frac{(n-1)a+b}{n+1} \ldots \frac{a+nb}{n+1} \quad \text{(by 80),}$$

$$\text{and } h_1 h_2 \ldots h_n = \frac{ab(n+1)}{nb+a} \cdot \frac{ab(n+1)}{(n-1)b+a} \ldots \frac{ab(n+1)}{b+na}, \quad \text{(by 81),}$$

$$\therefore a_1 a_2 \ldots a_n \times h_1 h_2 \ldots h_n = a^n b^n, \quad . \quad . \quad . \quad . \qquad \text{(ii).}$$

Hence from (i),

$$g_1 g_2 \ldots g_n = \sqrt{(a_1 a_2 \ldots a_n) \times (h_1 h_2 \ldots h_n)}.$$

Q.E.D.

(ii) Prove that any even number of terms of the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ \ldots$$

is equal to the latter half of these terms all taken positively.

Let $F(2x)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ \ldots\ -\frac{1}{2x}$, . . (i),

and $\phi(2x)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \ldots\ +\frac{1}{2x}$, . . (ii);

$\therefore\quad \phi(x)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \ldots\ +\frac{1}{x}$, . . (iii);

$$\therefore\ 2F(2x)=2\left\{1+\tfrac{1}{3}+\ \ldots\ +\frac{1}{2x-1}\right\}$$
$$-\left\{1+\tfrac{1}{2}+\tfrac{1}{3}+\ \ldots\ +\frac{1}{x}\right\},$$
$$=F(2x)+\phi(2x)-\phi(x);$$

$\therefore\quad F(2x)=\phi(2x)-\phi(x)$. Q.E.D.

(iii) The mth term of an A.P. is $3m+1$. Find the sum of n terms.

Since the mth term $=3m+1$,

we have by putting $m=1$, the first term $=3.1+1=4$,
" $m=2$, " second $=3.2+1=7$,
" $m=3$, " third $=3.3+1=10$;

$\therefore$ the series is $4+7+10+\ \ldots$

$$\therefore\ S=\{8+(n-1)3\}\frac{n}{2}=\frac{(3n+5)n}{2}.$$

(iv) a, b, c, are in H.P., prove $\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$, in H.P.

Since a, b, c, are in H.P.,

$\therefore\ \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, ... A.P.;

$\therefore\ \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$, . . . A.P.;

$\frac{b+c}{a}+1, \frac{c+a}{b}+1, \frac{a+b}{c}+1$, . . . A.P.;

$\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$, ... A.P.;

$$\therefore \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}, \quad . \quad . \quad \text{H.P.};$$

(v) The sum of m terms of an A.P. is to the sum of n terms in the ratio $m^2 : n^2$. Show that the mth term is to the nth in the ratio $(2m-1) : (2n-1)$.

For putting $m=1$ and $n=2$ we have by question

$$\frac{a}{2a+d}=\frac{1}{4}; \quad \therefore d=2a.$$

$\therefore$ The series is $a+3a+5a+ \ldots$

$\therefore$ The mth term $=(2m-1)a$,

and the nth term $=(2n-1)a$;

$\therefore$ the required result follows.

RATIO AND PROPORTION.

83. **Ratio is the mutual relation of two quantities of the same kind, the one to the other with respect to quotity.**

Hence $a : b$ and $\frac{a}{b}$ mean the same thing.

The first term of a ratio is called the antecedent, the second term the consequent. Thus the antecedent of a ratio is the N^r of a fraction, and the consequent of a ratio the D^r of a fraction.

A ratio is said to be of greater or less inequality, or of equality, according as the antecedent is greater than, less than, or equal to the consequent.

84. A ratio of greater inequality is diminished by adding the same quantity to each term of the ratio.

Let $\frac{a}{b}$ be a ratio of greater inequality, and let x be added to each term.

Then $\frac{a+x}{b+x}<\frac{a}{b}$,

if $ab+bx<ab+ax$,

if $bx<ax$,

if $b<a$, which it is by Hyp.

$\therefore$ etc.

Similarly, it may be shown that a ratio of less inequality is increased by adding the same quantity to each of its terms.

85. Theory of Incommensurables.

Two quantities are said to be incommensurable when their ratio cannot be found exactly; in other words, when there is no quantity, however small, which is exactly contained in both of them.

Geometrical illustration—

Let CD, AB be two incommensurable quantities, the former being divided into n equal parts, of which CF is one. Let each of these parts be denoted by x; $\therefore$ $CD=nx$.

Now since AB and CD are incommensurable, it follows that if we mark upon AB parts equal to CF beginning at A, we shall find that CF does not measure AB, but that beyond the last division Q there will be a remainder QB, and that such a remainder will exist however small we make x.

Now let $AQ=mx$; $\therefore$ AB lies between mx and $(m+1)x$. The ratio of the commensurable quantities AQ, CD can be found exactly $\left(\frac{m}{n}\right)$—that of the incommensurable quantities AB, CD cannot. All we know of it is that it lies between $\frac{m}{n}$ and $\frac{m+1}{n}$.

86. The ratio of a to b may, however, when these quantities are incommensurable, be approximated to as nearly as we please.

For as in the preceding section $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$;

$\therefore$ $\frac{m}{n}$, $\frac{a}{b}$, $\frac{m+1}{n}$ are in ascending order of magnitude;

$\therefore$ $\frac{a}{b}$ differs from $\frac{m}{n}$ by less than $\frac{1}{n}$, and by sufficiently increasing n and diminishing x, this may clearly be made as small as we please.

87. If $\frac{a}{b}$ and $\frac{c}{d}$, two incommensurable ratios, both lie between $\frac{m}{n}$ and $\frac{m+1}{n}$, they are equal.

For their difference $<\frac{1}{n}$ which can be made less than any assignable quantity.

88. Proportion is the equality of ratios.

Since ratios must involve quantities of the same kind, it follows that the first and second terms of a proportion must be of the same kind; as also the third and fourth terms—though, of course, it is not necessary that the first pair should be of the same kind as the second pair.

The usual definitions of proportion are—

Algebraical.—Four quantities are said to be proportional when the first is the same multiple or sub-multiple of the second as the third is of the fourth.

Geometrical.—Four quantities are said to be proportional when any equi-multiples of the first and third being taken, and likewise of the second and fourth, it is found that if the multiple of the first be greater than the multiple of the second, the multiple of the third is greater than that of the fourth; if equal, equal; and if less, less.

89. If four quantities be proportional according to the algebraical definition, so are they according to the G. definition.

By Hyp. $\frac{a}{b}=\frac{c}{d}$, a, b, c, d, being the four quantities,

and $\frac{m}{n}=\frac{m}{n}$;

$\therefore \frac{ma}{nb}=\frac{mc}{nd}$;

$\therefore$ if $ma>nb$, mc is $>nd$, if equal, equal; and if less, less. Therefore, etc.

90. If four quantities be proportional according to the geometrical definition, so are they according to the algebraical definition.

(i) Let $\frac{a}{b}$ and $\frac{c}{d}$ be commensurable fractions.

Now by Hyp. $ma >=$ or $< nb$ according as $mc >=$ or $< nd$.

Take m, n, such that $ma = nb$ (this can always be done);

$$\therefore\ mc = nd;$$

$$\therefore\ \frac{a}{b} = \frac{n}{m} = \frac{c}{d}.$$

(ii) Let $\frac{a}{b}$ and $\frac{c}{d}$ be incommensurable fractions.

Here we cannot take m and n, such that $ma = nb$; but we can take them such that

$$ma \text{ lies between } nb \text{ and } (n+1)b;$$

$$\therefore\ mc \quad \text{,,} \quad nd \text{ ,, } (n+1)d;$$

$$\therefore\ \frac{a}{b} \text{ and } \frac{c}{d} \text{ each lie between } \frac{n}{m} \text{ and } \frac{n+1}{m};$$

$$\therefore\ \frac{a}{b} = \frac{c}{d}, \qquad . \quad . \quad . \qquad (87).$$

VARIATION.

91. Defs.—(i) x is said to vary as y *directly*, when if x be changed to x', y is *consequently* changed to y', in such a way that

$$x : x' :: y : y'.$$

(ii) x is said to vary as y *inversely*, when if x be changed to x', y is *consequently* changed to y', in such a way that

$$x : x' :: \frac{1}{y} : \frac{1}{y'}.$$

(iii) x is said to vary as yz *jointly*, when if x be changed to x', y is *consequently* changed to y', and z to z', in such a way that

$$x : x' :: yz : y'z'.$$

In case (i) it appears that $\frac{x}{y}=\frac{x'}{y'}$,

(ii) „ $xy=x'y'$,

(iii) „ $\frac{x}{yz}=\frac{x'}{y'z'}$.

Hence we have the following simpler definitions :—

(i) x varies as y *directly*, when however x and y are changed their *ratio* is constant.

(ii) x varies as y *inversely*, when however x and y are changed their *product* is constant.

(iii) x varies as y and z jointly, when however much x, y, and z are changed, the ratio of the first to the product of the other two is constant.

Hence if $x \propto y$, $x=my$, where m is constant,

$x \propto \frac{1}{y}$, $xy=m$, „

$x \propto yz$, $x=myz$, „

It thus appears that variations may be turned into equations and worked as equations. This is of great importance, as it is the method of dealing with almost all questions in variations.

92. If $x \propto y$ when z is constant, and $\propto z$ when y is constant, then $x \propto yz$ when both vary.

For owing to the change of y into y', let x become x''; and owing to the change of z into z', let x'' become x';

$$\therefore x : x'' :: y : y',$$

$$\text{and } x'' : x' :: z : z';$$

$$\therefore xx'' : x''x' :: yz : y'z';$$

$$\therefore x : x' : yz : y'z';$$

$$\therefore x \propto yz.$$

93. The following examples will illustrate the manner of working variations:—

(i) $y^2 \propto x$ and when $x=a$, $y=2a$. Find the equation between x and y.

Since $y^2 \propto x$; $\therefore$ $y^2=mx$, where m is constant.
But $x=a$, $y=2a$, satisfy this; $\therefore$ $4a^2=m.a.$;
$\therefore$ $m=4a$; $\therefore$ the required equation is $y^2=4ax$.

(ii) y is equal to the sum of two quantities, one of which varies as x, and the other as x inversely; when $x=1$, $y=5$, and when $x=3$, $y=7$. Find the equation between x and y.

By question $y=mx+\frac{n}{x}$, where m and n are constant.

Hence $\quad 5=m+n.$

and $\quad 7=3m+\frac{n}{3}$; $\therefore$ $21=9m+n.$

Hence $16=8m$; $\therefore$ $m=2$, whence $n=3$,

and the required equation is

$$y=2x+\frac{3}{x}.$$

Generally the method is this: Write down the variation as an equation with undetermined constants. Contemporaneous values of the variables will be given, by means of which the constants may be determined. Then, finally, write down the equation with the determined instead of the undetermined constants.

(iii) The cost of each copy of a work, A, varies as the square root of the number of copies in the edition, and of another work, B, as the cube root of the number of copies in the edition. If an edition of a of the former and b of the latter be printed, the cost of two copies (one of each) is £c, and also d of the former cost as much as e of the latter. Find the cost of a copy of each in those editions.

By question,

cost of one copy of $A = m\sqrt{a}$, where m is constant,

" $\qquad B = n\sqrt[3]{b}$, where n is constant.

Hence $ma^{\frac{1}{2}} + nb^{\frac{1}{3}} = c$, ... (i),

and $\quad mda^{\frac{1}{2}} - neb^{\frac{1}{3}} = 0$, ... (ii);

$\therefore$ multiplying (i) by e and adding (ii),

$$ma^{\frac{1}{2}}(d+e) = ce\,; \quad \therefore\ m = \frac{ce}{a^{\frac{1}{2}}(d+e)}.$$

Again, multiplying (i) by d and subtracting,

$$nb^{\frac{1}{3}}(d+e) = cd\,; \quad n = \frac{cd}{b^{\frac{1}{3}}(d+e)}\,;$$

$$\therefore \text{ the cost of one copy of } A = \frac{ce}{a^{\frac{1}{2}}(d+e)}a^{\frac{1}{2}} = \frac{ce}{d+e},$$

$$\text{and} \quad \text{"} \quad B = \frac{cd}{b^{\frac{1}{3}}(d+e)}.b^{\frac{1}{3}} = \frac{cd}{d+e}.$$

PERMUTATIONS AND COMBINATIONS.

94. By the *permutations* of any number of things is meant the different arrangements that can be made of them, with reference to the order in which they appear.

Thus the number of permutations of *three things taken two together* is six, viz., *ab*, *ba*, *ac*, *ca*, *bc*, *cb*.

By the *combinations* of any number of things is meant, the different arrangements which can be made of them, without reference to the order in which they appear.

Thus the number of combinations of *three things*, *a*, *b*, *c*, *two together*, is three, viz., *ab*, *bc*, *ca*.

It will be seen from this that *ab* and *ba* are different permutations; but inasmuch as they involve only the same letters, they are the *same combination.* To make a different combination there must be at least one letter different.

The number of permutations of n things r togethe
by ${}^nP_r$—the number of combinations of n things r
${}^nC_r$.

95. *To find* ${}^nP_r$.

Let there be n quantities $a, b, c, \ldots$ Remove one of them, a, and the number of permutations which we can form of the remaining $(n-1)$ quantities taken $(r-1)$ together is denoted by ${}^{n-1}P_{r-1}$. And before each of these we can place a, so obtaining ${}^{n-1}P_{r-1}$ permutations of n things r together, *in which a stands first.* Similarly there will ${}^{n-1}P_{r-1}$ in which b stands first, and so for *each* of the n quantities.

Hence ${}^nP_r = n \times {}^{n-1}P_{r-1}$,

so ${}^{n-1}P_{r-1} = (n-1) \times {}^{n-2}P_{r-2}$ $\left\{\begin{array}{l}\text{by writing } n-1,\ r-1 \text{ for} \\ \quad n,\ r \text{ respectively,}\end{array}\right.$

and ${}^{n-2}P_{r-2} = (n-2) \times {}^{n-3}P_{r-3}$,

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$${}^{n-r+2}P_2 = (n-r+2) \times {}^{n-r+1}P_1.$$

$\therefore$ by multiplication, and remembering that ${}^{n-r+1}P_1 = (n-r+1)$ obviously, we have

$${}^nP_r = n(n-1)(n-2) \ldots (n-r+1).$$

Hence we see that ${}^nP_2 = n(n-1)$, ${}^nP_3 = n(n-1)(n-2)$, and so on. Also ${}^nP_n = n.\overline{n-1} \ldots 3.2.1 = \underline{|n}$.

The symbol $\underline{|n}$, denoting the product of the first n natural numbers, is read "*factorial n.*"

96. *To find* ${}^nC_r$.

Consider the n quantities $a, b, c, \ldots$ Remove one of them, a, then ${}^{n-1}C_{r-1}$ denotes the number of combinations which can be formed of the remaining $(n-1)$ quantities $(r-1)$ together, and in each of these a may be placed. Therefore the number of combinations of *n things r together* in which a appears is ${}^{n-1}C_{r-1}$.

Similarly the number of combinations in which b appears is ${}^{n-1}C_{r-1}$, and so for each of the n quantities; *but since each*

combination of r things is thus repeated r times, it is plain that the number of combinations of n things r together is thus $\frac{n}{r}\times{}^{n-1}C_{r-1}$.

$$
\begin{aligned}
\text{Therefore } {}^{n}C_{r} &= \frac{n}{r} \times {}^{n-1}C_{r-1}, \\
{}^{n-1}C_{r-1} &= \frac{n-1}{r-1} \times {}^{n-2}C_{r-2} \left\{ \begin{array}{l} \text{writing } \overline{n-1}| \text{ for } n \\ \text{and } \overline{r-1}| \text{ for } r, \end{array} \right. \\
\text{so } {}^{n-2}C_{r-2} &= \frac{n-2}{r-2} \times {}^{n-3}C_{r-3}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
{}^{n-r+2}C_{2} &= \frac{n-r+2}{2} \times {}^{n-r+1}C_{1};
\end{aligned}
$$

$\therefore$ by multiplication

$$
{}^{n}C_{r} = \frac{n.\overline{n-1}\ \ldots\ \overline{n-r+2}}{\underline{|r}}\ {}^{n-r+1}C_{1},
$$

but ${}^{n-r+1}C_{1} = n-r+1$, obviously;

$$
\therefore\ {}^{n}C_{r} = \frac{n.\overline{n-1}\ \ldots\ \overline{n-r+1}}{\underline{|r}} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.
$$

This proof as independent of ${}^{n}P_{r}$ is the only practical one to write in examinations. If, however, the *value of ${}^{n}P_{r}$ be assumed* a very simple consideration will show that ${}^{n}C_{r}=\frac{{}^{n}P_{r}}{\underline{|r}}$, and so the required value for ${}^{n}C_{r}$ is found. The student should see this for himself.

97. ${}^{n}C_{r}={}^{n}C_{n-r}$.

For every different group of r things, there must be a corresponding different group of $(n-r)$ things;

$$
\therefore\ {}^{n}C_{r}={}^{n}C_{n-r}.
$$

When r is more than half n this theorem may be advantageously used in finding ${}^{n}C_{r}$.

For instance $^{10}C_8=\dfrac{10.9.8.7.6.5.4.3}{1.2.3.4.5.6.7.8}$ in which fraction 3.4.5.6.7.8 cancels, and the result $=\dfrac{10.9}{1.2}$.

But if we remember $^{10}C_8={}^{10}C_2=\dfrac{10.9}{1.2}$, the factors which cancel need *not be written down*, and so labour is saved.

Again, since $^nC_r={}^nC_{n-r}$;

$$\therefore\ {}^nC_0={}^nC_{n-0}={}^nC_n=1.$$

This result and also $\underline{|0}=1$ (which may be proved as follows), should be remembered,

$$\underline{|n}=n\underline{|n-1}\,; \qquad \therefore\ \underline{|1}=1\underline{|1-1}\,; \qquad \therefore\ \underline{|0}=1.$$

98. *To find the number of permutations which can be formed out of n things taken all together, some of which are the same.*

Let p of the n things be a, q of them b, r of them c, and so on, and let P be the required number of permutations.

Change the p a's into different letters, and then *each one of the P permutations* will give rise to $\underline{|p}$ permutations, and thus we shall have in all $P\underline{|p}$ permutations. Similarly by changing the q b's into different letters we shall obtain $P\underline{|p}\,\underline{|q}$ permutations, until at last when we have made *all* the letters different we obtain $P\underline{|p}\,\underline{|q}\,\underline{|r}\ .\ .\ .$ permutations. But when *all* the letters are different the number of permutations which can be made of them taken all together is $\underline{|n}$;

$$\therefore\ P\underline{|p}\,\underline{|q}\,\underline{|r}\ .\ .\ .\ =\underline{|n}\,;$$

$$\therefore\ P=\frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}.$$

99. To find the number of combinations which can be made of two sets of things, one of which contains *m* and the other *n*, one being taken out of each set for each combination.

Clearly each of the m things will go with each of the n things, so forming mn different combinations.

So also if another set containing s things be introduced, the number of combinations that can be formed from the three sets, one being taken from each set, for each combination is mns. The same reasoning may be still further applied to more than three sets.

100. **To find the number of combinations which can be made out of two sets of things, one of which contains m things and the other n, p being taken out of the first and q out of the second for each combination.**

Since the number of different parcels p together, taken from the first set is ${}^mC_p$, and q together taken from the second set is ${}^nC_q$, and since each of the former will go with each of the latter, the number of combinations required is ${}^mC_p \times {}^nC_q$.

101. **To find what value of r makes ${}^nC_r$ a maximum.**

$${}^nC_r = \frac{n.\overline{n-1}\ .\ .\ .\ \overline{n-r+1}}{\underline{|r}},$$

and $${}^nC_{r+1} = \frac{n.\overline{n-1}\ .\ .\ .\ \overline{n-r}}{\underline{|r+1}},$$

$$\therefore\ {}^nC_{r+1} = {}^nC_r \times \frac{n-r}{r+1};$$

$\therefore$ ${}^nC_r$ will be a maximum when $\frac{n-r}{r+1}$ *is first* <1;

$\therefore$ when $n-r$ is first $<r+1$,

" $n-1$ " $<2r$,

" r " $>\frac{n-1}{2}$.

Note.—If n be an *even* number $\frac{n-1}{2}$ will be fractional, and the *integer* next greater than this will be the value of r which makes ${}^nC_r$ a maximum; but if n be odd $\frac{n-1}{2}$ is integral. Then that integer and the integer next greater than it will give two values for r, whence two maxima will be found.

Example.—What value of r makes ${}^6C_r$ a maximum?

$$\frac{n-1}{2}=\frac{5}{2}=2\tfrac{1}{2}\,; \quad \therefore\ r=3 \text{ will make } {}^6C_r \text{ a maximum.}$$

To make this still clearer to the mind of the student, we show the values of ${}^6C_1 {}^6C_2$ . . .

$$\begin{aligned}
{}^6C_1 &= \frac{6}{1} &&= 6,\\
{}^6C_2 &= \frac{6}{1}\times\frac{5}{2} &&= 15,\\
{}^6C_3 &= \frac{6.5}{1.2}\times\frac{4}{3} &&= 20,\\
{}^6C_4 &= \frac{6.5.4}{1.2.3}\times\frac{3}{4} &&= 15,\\
{}^6C_5 &= \frac{6.5.4.3}{1.2.3.4}\times\frac{2}{5} &&= 6,\\
{}^6C_6 &= \frac{6.5.4.3.2}{1.2.3.4.5}\times\frac{1}{6} &&= 1.
\end{aligned}$$

Each of these combinations it will be seen is formed by multiplying the preceding one by *a fraction*, and that this fraction continually decreases. As soon as it becomes <1, that is in ${}^6C_4$, where it is $\frac{3}{4}$, we have ${}^6C_4$ a proper fraction of ${}^6C_3$, and thus ${}^6C_3$ is the maximum required.

Example 2.—What value of r makes ${}^5C_r$ a maximum?

Here $\frac{n-1}{2}=2\,;\quad \therefore\ r=2$, or $r=3$ will give maxima.

$${}^5C_1=\frac{5}{1}, \qquad {}^5C_2=\frac{5.|4}{1.|2}, \qquad {}^5C_3=\frac{5.4\,|3}{1.2\,|3}, \qquad {}^5C_4=\frac{5.4.3\,|.2}{1.2.3\,|.4}.$$

Here ${}^5C_2={}^5C_3=10$, and after ${}^5C_3$ the fraction mentioned above becomes <1.

BINOMIAL THEOREM.

102. Sir Isaac Newton discovered that

$$(x+a)^n=x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+\ .\ .\ .\ +{}^nC_na^n.$$

Proof.—

(i) **Let n be a positive integer.**

By writing $-a$, $-b$, $-c$ . . . for $a_1a_2a_3$ . . . in 59 we have

$$(x+a)(x+b)(x+c)\ldots \text{ to } n \text{ factors}$$
$$=x^n+(a+b+\ldots)x^{n-1}+(ab+bc+\ldots)x^{n-2}+\ldots$$
$$+abc\ldots \text{ to } n \text{ factors.}$$

Let $b=c=d=\ .\ .\ .\ =a$.

The left-hand side becomes $(x+a)^n$.

The coefficients of the right-hand side are 1,

$$(a+a+a\ldots\text{to } {}^nC_1 \text{ terms})={}^nC_1a,$$
$$(a^2+a^2+\ldots\text{to } {}^nC_2 \text{ terms})={}^nC_2a^2,$$
$$.\quad .\quad .\quad .\quad .\quad .\quad .$$

the last term being ${}^nC_na^n$.

Hence $(x+a)^n=x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+\ldots+{}^nC_na^n$.

Notice; n denotes the *number of factors*, and therefore must be a positive integer.

This proof therefore will not hold when n is fractional or negative. In these cases we give the following proof (Euler's):—

(ii) Let $f(m)=1+mx+\frac{m(m-1)}{1.2}x^2+\ .\ .\ .$ **for all values of** m;

$$\therefore\quad f(n)=1+nx+\frac{n(n-1)}{1.2}x^2+\ \ .\ .\ .$$

$$\therefore f(m)f(n)=1+ax+bx^2+\ .\ .\ .$$

where a, b, c are quantities involving m and n.

Now although by changing the values of m and n we change the *values* of $a, b, c,$ . . . still we cannot change the *form* in which m and n enter into $a, b, c,$ . . . Hence if we can find this form in

any one case it will be the form universally. Take therefore *the simplest case*, viz., when m and n are *positive integers.* Now in this case $f(m)=(1+x)^m$ and $f(n)=(1+x)^n$;

$$\therefore f(m).f(n)=(1+x)^{m+n}$$

$$=1+\frac{m+n}{1}.x+\frac{(m+n)(m+n-1)}{1.2}x^2+\ \ldots$$

The right-hand side is therefore the form *universally.*

But this right-hand side $=f(m+n)$;

$$\therefore\ f(m)f(n)=f(m+n) \text{ universally};$$

$$\text{so } f(m)f(n)f(p)=f(m+n)f(p)=f(m+n+p),$$

and so whatever be the number of factors.

Now let each factor $=f\left(\frac{h}{k}\right)$ and let their number be k, h and k being positive integers, so that $\frac{h}{k}$ is a positive fraction.

Then

$$f\left(\frac{h}{k}\right).\ f\left(\frac{h}{k}\right)\ldots \text{to } k \text{ factors,} =f\left(\frac{h}{k}+\frac{h}{k}+\ldots\text{to } k \text{ terms}\right);$$

$$\therefore\ \left[f\left(\frac{h}{k}\right)\right]^k=f(h)=(1+x)^h \text{ for } h \text{ is a positive integer};$$

$$\therefore\ (1+x)^{\frac{h}{k}}=f\left(\frac{h}{k}\right)$$

$$=1+\frac{h}{k}.x+\frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1.2}x^2+\ \ldots$$

which proves the Theorem for *a fractional index.*

Again, since $f(m-n)f(n)=f(m-n+n)=f(m)$;

$$\therefore\ f(m-n)=\frac{f(m)}{f(n)},$$

$$\text{but } f(m-n)=f(m)f(-n),$$

$$\therefore\ f(m)f(-n)=\frac{f(m)}{f(n)},$$

$$\therefore\ f(-n)=\frac{1}{f(n)}=\frac{1}{(1+x)^n}=(1+x)^{-n};$$

$$\therefore\ (1+x)^{-n}=1+\frac{-n}{1}x+\frac{-n(-n-1)}{1.2}x^2+\ .\ .\ .$$

which proves the theorem for a *negative* index.

Notice.—From the above it follows that $f(n)f(-n)=1$;

$$\therefore\ f(n-n)=1;\quad \therefore\ f(0)=1.$$

103. The student may notice that in the proof for a positive index we took the binomial $(x+a)^n$—but that in Euler's proof for a fractional and negative index we took the binomial $(1+x)^n$, and he may possibly be inclined to suppose, that if we now assume the theorem for $(a+x)^n$ when n is not a positive integer, that we are arguing from particular to general. If he experience this difficulty he should observe that if

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ .\ .\ .$$

Then on writing $\frac{x}{a}$ for x,

$$\left(1+\frac{x}{a}\right)^n=1+{}^nC_1\cdot\frac{x}{a}+{}^nC_2\frac{x^2}{a^2}+\ .\ .\ .$$

$$\therefore\ \frac{(a+x)^n}{a^n}=1+{}^nC_1\cdot\frac{x}{a}+{}^nC_2\frac{x^2}{a^2}+\ .\ .\ .$$

$$\therefore\ (a+x)^n=a_n+{}^nC_1a^{n-1}x+{}^nC_2a^{n-2}x^2+\ .\ .\ .$$

So that the apparently simpler form includes the other also.

104. It is thus established for all values of n that

$$(1+x)^n=1+\frac{n}{1}x+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ \text{(i)};$$

$$\therefore\ (1-x)^n=1-\frac{n}{1}x+\frac{n(n-1)}{1.2}x^2$$

$$-\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ \text{by writing } -x \text{ for } x,\qquad \text{(ii)},$$

$$(1+x)^{-n}=1-\frac{n}{1}x+\frac{n(n+1)}{1.2}x^2$$

$$-\frac{n(n+1)(n+2)}{1.2.3}x^3+\dots \left\{\begin{array}{c}\text{by writing } -n \text{ for} \\ n \text{ in (i),}\end{array}\right\} \text{(iii)},$$

and $$(1-x)^{-n}=1+\frac{n}{1}x+\frac{n(n+1)}{1.2}x^2$$

$$+\frac{n(n+1)(n+2)}{1.2.3}x^3+\dots \left\{\begin{array}{c}\text{by writing } -x \text{ for} \\ x \text{ in (iii),}\end{array}\right\} \text{(iv)}.$$

The results are easily remembered, thus—

If the signs of n and x are the same, the signs of all the terms are $+$.

If they are different, the signs are alternately $+$ *and* $-$.

If the index is $+$, *the factors in the numerators decrease in order.*

If the index is $-$, *the factors increase in order.*

Also if n be a *positive integer* we shall at last obtain a factor in the N^r, $n-n$, which$=0$.

Therefore the series will terminate.

This can never be the case when n is *negative or fractional.* Hence in these instances the series will go on *ad infinitum*, and thus **when x is >1 the expansions will be arithmetically unintelligible.** To this point, however, the student should recur after reading the chapter on the convergency or divergency of series.

105. If we consider the identity

$$(a+x)^n=a^n+{}^nC_1a^{n-1}x+{}^nC_2a^{n-2}x^2+\dots+{}^nC_{n-1}ax^{n-1}+{}^nC_nx^n.$$

It is evident—

(i) That the *general* or $(r+1)$th term $={}^nC_ra^{n-r}x^r$.

(ii) That the number of terms $=(n+1)$.

For the index of x in any term shows how many terms precede it. Therefore since the index in the last term is n, there are n terms preceding the last, and thus altogether there are $(n+1)$ terms.

(iii) The coefficients of terms equidistant from the beginning and end are equal.

For the coefficient of the $(r+1)$th term from the beginning is ${}^nC_r$, and the coefficient of the $(r+1)$th from the end is ${}^nC_{n-r}$, and these are equal by Art. 97.

106. To find the *total number* of combinations of n things; in other words, to find the sum of all the coefficients in an expanded binomial,

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ .\ .\ .\ {}^nC_nx^n.$$

Put $x=1$, and we have

$${}^nC_1+{}^nC_2+\ .\ .\ .\ +{}^nC_n=2^n-1.$$

107. **To find the numerically greatest term in $(1+x)^n$, x being considered positive.**

(i) Let n be a positive integer.

Now the $(r+1)$th term$={}^nC_rx^r$, and the rth term$={}^nC_{r-1}x^{r-1}$;

$$\therefore \text{ the } (r+1)\text{th term} = \text{the } r\text{th} \times \frac{n-r+1}{r},$$

$$= \text{the } r\text{th} \times \left(\frac{n+1}{r}-1\right)x.$$

Now as r increases this multiplier diminishes, and the rth term will be a maximum when $\left(\frac{n+1}{r}-1\right)x$ is first <1, which condition reduces to, when r is first $>(n+1)\frac{x}{1+x}$.

Again, if $r=(n+1)\frac{x}{1+x}$, we should have $\left(\frac{n+1}{r}-1\right)x=1$, and therefore there would be two equal maxima, viz., the rth and the $(r+1)$th terms.

(ii) Let n be fractional.

Then though the multiplier above decreases with r, it can never be numerically $<x$, even when *r is made infinitely great.* Hence unless x be <1 no maximum exists.

If, however, x be <1 the previous result holds.

(iii) Let n be negative and $=-m$, say.

Then the $(r+1)$th term $=$ the rth $\times\dfrac{m+r-1}{r}.x$ *numerically*,

$$=\text{the } r\text{th}\times\left(\frac{m-1}{r}+1\right)x \quad ,,$$

Here, too, though the multiplier decreases with r, it can never be made $<x$, even when r is made infinitely great. Hence unless x is <1 no maximum term exists. If, however, x be <1, the rth term will be a maximum when $\left(\dfrac{m-1}{r}+1\right)x$ is first <1, which becomes on reduction, when

$$r \text{ is first greater than } (m-1)\frac{x}{1-x}.$$

That is when r is first greater than $(n+1)\dfrac{x}{x-1}$.

If this be integral as before, there will be two maxima.

Example. (i) What term of $(1+\frac{1}{2})^{\frac{3}{2}}$ is a maximum?

$$(n+1)\frac{x}{1+x}=\frac{5}{2}\cdot\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{5}{2}\cdot\frac{1}{3}=\frac{5}{6};$$

$\therefore$ the first term is a maximum.

(ii) Which term of $(1+\frac{9}{10})^{-2}$ is a maximum?

$$(n+1)\frac{x}{x-1}=-1\cdot\frac{\frac{9}{10}}{-\frac{1}{10}}=9;$$

$\therefore$ the ninth and tenth terms are maxima.

108. **To find the number of homogeneous products of r dimensions which can be made with the n things $a, b, c, \ldots$ and their powers.**

By actual Division,

$$\frac{1}{1-ax}=1+ax+a^2x^2+\ \ldots$$

$$\frac{1}{1-bx}=1+bx+b^2x^2+ \ . \ . \ .$$

$$\frac{1}{1-cx}=1+cx+c^2x^2+ \ . \ . \ .$$

$$. \quad . \quad . \quad . \quad . \quad .$$

There being n rows. Hence by Multiplication,

$$\frac{1}{1-ax}\cdot\frac{1}{1-bx}\cdot\frac{1}{1-cx}\ldots=1-(a+b+c+\ldots)x+(ab+a^2+\ldots)x^2$$
$$+(a^3+a^2b+abc+\ldots)x^3+\ldots$$

Therefore the coefficient of x^r on the right-hand side is the sum of the homogeneous products of r dimensions formed with the given n things and their powers.

Now let $a=b=c= \ . \ . \ . \ =1$. This coefficient then becomes the *number* of such products; and since the left-hand side becomes

$$(1-x)^{-n}=1+\frac{n}{1}x+\frac{n(n+1)}{\lfloor 2}x^2+\ldots$$
$$+\frac{n(n+1)\ldots(n+r-1)}{\lfloor r}x^r+\ldots$$

we have by equating coefficients of x^r,

the required number $=\dfrac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}$.

109. To approximate to the root of a number by the Binomial Theorem.

Example. Find $\sqrt[5]{34}$.

$$\sqrt[5]{34}=\sqrt[5]{32+2}=\sqrt[5]{32\left(1+\frac{1}{2^4}\right)}=2\left(1+\frac{1}{2^4}\right)^{\frac{1}{5}}$$

$$=2\left\{1+\frac{1}{5}\cdot\frac{1}{2^4}+ \ . \ . \ . \ \right\},$$

and by taking a few terms of this we shall obtain a tolerably close approximation.

110. To find the remainder after n terms of

(i) $(1-x)^{-1}$. (ii) $(1-x)^{-2}$.

(i) $(1-x)^{-1}=(1+x+x^2+\ldots+x^{n-1})+x^n+x^{n+1}+\ldots ad\ inf.$

Hence if R denote the remainder

$$R=\frac{x^n}{1-x}, \quad . \quad . \quad . \quad \text{(Art. 76)}.$$

(ii) $(1-x)^{-2}=(1+2x+3x^2+\ .\ .\ .\ +nx^{n-1})+R.$

Let $S=1+2x+3x^2+\ .\ .\ .\ +nx^{n-1}$;

$\therefore\ xS=x+2x^2+\ .\ .\ .\ +(n-1)x^{n-1}+nx^n$;

$\therefore\ S(1-x)=(1+x+\ .\ .\ .\ +x^{n-1})-nx^n$;

$$\therefore\ S=\frac{1}{1-x}\left\{\frac{1-x^n}{1-x}-nx^n\right\};$$

$$\therefore\ R=\frac{1}{1-x}\left\{\frac{1}{1-x}-\frac{1-x^n}{1-x}+nx^n\right\},$$

$$=\frac{1}{1-x}\left\{\frac{x^n}{1-x}+nx^n\right\}.$$

111. It should be carefully remembered that

$(1-x)^{-1}=1+x+x^2+x^3+\ .\ .\ .\ ad\ inf.$

$(1-x)^{-2}=1+2x+3x^2+4x^3+\ .\ .\ .$

$(1-x)^{-3}=1+3x+\frac{3.4}{1.2}x^2+\frac{4.5}{1.2}x^3+\frac{5.6}{1.2}x^4+\ .\ .\ .$

112. The following are a few problems which will be found suggestive:—

(i) Prove

$$(1+x)^{2n}=(1+x)^n+\frac{n}{1}(1+x)^{n-1}.x+\frac{n(n+1)}{1.2}(1+x)^{n-2}x^2+\ldots$$

The right-hand side

$$=(1+x)^n\left\{1+\frac{n}{1}\cdot\left(\frac{x}{1+x}\right)+\frac{n(n+1)}{1.2}\left(\frac{x}{1+x}\right)^2+\dots\right\},$$

$$=(1+x)^n\left\{1-\frac{x}{1+x}\right\}^{-n},$$

$$=(1+x)^n\left(\frac{1+x-x}{1+x}\right)^{-n},$$

$$=(1+x)^n(1+x)^n=(1+x)^{2n}.$$

(ii) $1, \alpha, \beta, \gamma, \dots$ are the coefficients taken in order, in the expansion of $(a+b)^n$; prove that

$$1+\frac{\alpha}{2}+\frac{\beta}{3}+\dots=\frac{2^{n+1}-1}{n+1},$$

$$1+\frac{\alpha}{2}+\frac{\beta}{3}+\dots=1+\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}+\frac{n.(n-1)(n-2)}{1.2.3.4}+\dots$$

$$=\frac{1}{n+1}\left\{(n+1)+\frac{(n+1)n}{\lfloor 2}+\frac{(n+1)n(n-1)}{\lfloor 3}+\text{ etc. }\right\},$$

$$=\frac{1}{n+1}\left\{-1+(1+1)^{n+1}\right\}=\frac{2^{n+1}-1}{n+1}.$$

(iii) Find $1+x+\left(\frac{1}{n}+1\right)\frac{x^2}{2}+\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+\frac{1}{2}\right)\frac{x^3}{3}$

$$+\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+\frac{1}{2}\right)\left(\frac{1}{n}+\frac{1}{3}\right)\frac{x^4}{4}+\dots$$

ad inf. the series being convergent.

The series may be written

$$1+x+\frac{n+1}{2}\cdot\frac{x^2}{n}+\frac{(n+1)(n+2)x^3}{2.3.n^2}+\frac{(n+1)(n+2)(n+3)x^4}{2.3.4.n^3}+\dots$$

$$=1+n\left(\frac{x}{n}\right)+\frac{n(n+1)}{\lfloor 2}\left(\frac{x}{n}\right)^2+\frac{n(n+1)(n+2)}{\lfloor 3}\left(\frac{x}{n}\right)^3+\dots$$

$$=\left(1-\frac{x}{n}\right)^{-n}.$$

(iv) *If n be an even integer, prove*

$$1-n+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}+\ldots \textit{ to } \frac{n}{2} \textit{ terms,}$$

$$=\frac{(-1)^{\frac{n}{2}+1}}{2}\cdot\frac{n(n-1)\ \ldots\ \left(\frac{n}{2}+1\right)}{\left\lfloor\frac{n}{2}\right.}.$$

The sum to $\frac{n}{2}$ terms $=\frac{1}{2}$ {the sum of all the $(n+1)$ terms}.

$-\frac{1}{2}$ {the middle or $\left(\frac{n}{2}+1\right)$th term}.

Now the sum of all the $(n+1)$ terms $=(1-1)^n=0$,

and the $\left(\frac{n}{2}+1\right)$th term is $(-1)^{\frac{n}{2}}\cdot\dfrac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{\left\lfloor\frac{n}{2}\right.}$,

and $-\frac{1}{2}$ of this $=\dfrac{(-1)^{\frac{n}{2}+1}}{2}\cdot\dfrac{n(n+1)\ldots\left(\frac{n}{2}+1\right)}{\left\lfloor\frac{n}{2}\right.}$.

Therefore, etc.

(v) Prove $1+\frac{2n}{3}+\frac{2n(2n+2)}{3.6}+\frac{2n(2n+2)(2n+4)}{3.6.9}+\ldots$

$$=2^n\left\{1+\frac{n}{3}+\frac{n(n+1)}{3.6}+\frac{n(n+1)(n+2)}{3.6.9}+\ldots\right\}.$$

The left-hand side

$$=1+\frac{n}{1}\cdot\frac{2}{3}+\frac{n.\overline{n+1}}{1.2}\cdot\left(\frac{2}{3}\right)^2+\frac{n.(n+1)(n+2)}{1.2.3}\left(\frac{2}{3}\right)^3+\ldots$$

$$=\left(1-\frac{2}{3}\right)^{-n}=\left(\frac{1}{3}\right)^{-n}=3^n=2^n\left(\frac{3}{2}\right)^n=2^n\left(\frac{2}{3}\right)^{-n}$$

$$=2^n\left(1-\frac{1}{3}\right)^{-n}=2^n\left\{1+\frac{n}{1}\cdot\frac{1}{3}+\frac{n(n+1)}{1.2}\,\frac{1}{3^2}\right.$$
$$\left.+\frac{n(n+1)(n+2)}{1.2.3}\,\frac{1}{3^3}+\dots\right\}$$
$$=2^n\left\{1+\frac{n}{3}+\frac{n(n+1)}{3.6}+\frac{n(n+1)(n+2)}{3.6.9}+\dots\right\}.$$

(vi) Find $1+\frac{1}{3}+\frac{1.3}{1.2.9}+\frac{1.3.5}{1.2.3.27}+\dots$ *ad inf.*

This series

$$=1+\frac{1}{2}\left(\frac{2}{3}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{1.2}\left(\frac{2}{3}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)}{1.2.3}\left(\frac{2}{3}\right)^3+\dots$$
$$=\left(1-\frac{2}{3}\right)^{-\frac{1}{2}}=\left(\frac{1}{3}\right)^{-\frac{1}{2}}=3^{\frac{1}{2}}=\sqrt{3}.$$

(vii) Prove the sum (S) of the first $(r+1)$ coefficients in the expansion of $(1-x)^{-n}=\dfrac{\lfloor n+r}{\lfloor n\,\lfloor r}$.

$$(1-x)^{-n}=1+\frac{n}{1}\cdot x+\frac{n(n+1)}{1.2}x^2+\dots$$
$$+\frac{n(n+1)\dots(n+r-1)}{\lfloor r}x^r+\dots$$
$$(1-x)^{-1}=1+x+x^2+\dots+x^r+\dots$$

Multiply and equate coefficients of x^r.

That on the right-hand side is S.

That on the left is the coefficient of x^r in the expansion of $(1-x)^{-(n+1)}$.

But this is $\dfrac{(n+1)(n+2)\dots(n+r)}{\lfloor r}$;

$$\therefore\ S=\frac{(n+1)(n+2)\dots(n+r)}{\lfloor r},$$
$$=\frac{\lfloor n.(n+1)(n+2)\dots(n+r)}{\lfloor n.\lfloor r},$$
$$=\frac{\lfloor n+r}{\lfloor n\,\lfloor r}.$$

(viii) If n be a multiple of 3, prove that

$$1-(n-1)+\frac{(n-2)(n-3)}{\underline{|2}}-\frac{(n-3)(n-4)(n-5)}{\underline{|3}}+\text{ etc. } - =(-1)^n.$$

$$\left.\begin{array}{llll} 1 \text{ is the coefficient of } x^n \text{ in the expansion of} & & & (1-x)^{-1} \\ -(n-1) & ,, & ,, & -x^2(1-x)^{-2} \\ \dfrac{(n-2)(n-3)}{\underline{|2}} & ,, & ,, & x^4(1-x)^{-3} \end{array}\right\} \text{see Art. 111.}$$

. . . .

Hence if we denote the series given by S, we have S = the coefficient of x^n in

$$\frac{1}{1-x}-\frac{x^2}{(1-x)^2}+\frac{x^4}{(1-x)^3}-\ \ldots$$

But this latter series $=\dfrac{\dfrac{1}{1-x}}{1+\dfrac{x^2}{1-x}}=\dfrac{1}{1-x+x^2}=\dfrac{1+x}{1+x^3}$;

$\therefore\ S=$the coefficient of x^n in $(1+x)(1+x^3)^{-1}$

$$=(1+x)(1-x^3+x^6-\ \ldots),$$

and n is a multiple of 3,

$$S=(-1)^n.$$

Note.—If n be of the form $3m+1$, . . $S=(-1)^{n+1}$,
and if n be of the form $3m+2$, . . $S=0$.

(ix) Prove that

$$2^n-(n-1)2^{n-2}+\frac{(n-3)(n-2)}{1.2}2^{n-4}-\ldots=n+1.$$

$$\begin{array}{llll} 2^n \text{ is the coefficient of } x^n \text{ in the expansion of} & & & 2^n(1-x)^{-1}, \\ -(n-1)2^{n-2} & ,, & ,, & -2^{n-2}(1-x)^{-2}x^2, \\ \dfrac{(n-3)(n-2)}{1.2} & ,, & ,, & 2^{n-4}(1-x)^{-3}x^4, \end{array}$$

. . . .

Hence if we denote the given series by S, we have

$$S=\text{the coefficient of } x^n \text{ in } \frac{2^n}{1-x}-\frac{2^{n-2}x^2}{(1-x)^2}+\frac{2^{n-4}x^4}{(1-x)^3}-\ldots$$

But this latter series $=\dfrac{\dfrac{2^n}{1-x}}{1+\dfrac{x^2}{2^2(1-x)}}=\dfrac{2^{n+2}}{(2-x)^2}=2^{n+2}(2-x)^{-2}$

$$=2^n\left(1-\frac{x}{2}\right)^{-2},$$

$$=2^n\left\{1+2\left(\frac{x}{2}\right)+3\left(\frac{x}{2}\right)^2+4\left(\frac{x}{2}\right)^3+\ldots\right\},$$

and the coefficient of x^n in this

$$=2^n.\frac{n+1}{2^n}=(n+1),$$

$$\therefore\ S=n+1.$$

(x) Prove

$$(p+q)^n-(n-1)(p+q)^{n-2}pq$$
$$+\frac{(n-3)(n-2)}{1.2}(p+q)^{n-4}p^2q^2-\ldots=\frac{p^{n+1}-q^{n+1}}{p-q}.$$

Let S denote the given series, and for shortness denote $(p+q)$ by k.

Hence as in the two preceding examples,
S is the coefficient of x^n in the series

$$\frac{k^n}{1-x}-\frac{k^{n-2}.pq.x^2}{(1-x)^2}+\frac{k^{n-4}.p^2q^2.x^4}{(1-x)^3}-\ \ldots$$

But this series

$$=\frac{\dfrac{k^n}{1-x}}{1+\dfrac{pqx^2}{k^2(1-x)}}=\frac{k^{n+2}}{k^2(1-x)+pqx^2}=\frac{k^n}{(1-x)+\dfrac{pqx^2}{k^2}},$$

$$=\frac{k^n}{\left(1-\dfrac{px}{k}\right)\left(1-\dfrac{qx}{k}\right)},$$

$$=k^n\left\{\frac{1}{\left(1-\frac{px}{k}\right)\left(1-\frac{qx}{k}\right)}\right\},$$

$$=\frac{k^n}{p-q}\left\{\frac{p}{1-\frac{px}{k}}-\frac{q}{1-\frac{qx}{k}}\right\},$$

$$=\frac{k^n}{p-q}\left\{p\left(1-\frac{px}{k}\right)^{-1}-q.\left(1-\frac{qx}{k}\right)^{-1}\right\},$$

and the coefficient of x^n in this

$$=\frac{k^n}{p-q}\left\{p.\frac{p^n}{k^n}-q.\frac{q^n}{k^n}\right\},$$

$$=\frac{p^{n+1}-q^{n+1}}{p-q};$$

$$\therefore\ S=\frac{p^{n+1}-q^{n+1}}{p-q}.$$

(xi) *If p be nearly equal to q, show that*

$$\sqrt[3]{\frac{p}{q}}=\frac{2p+q}{p+2q}\text{ nearly.}$$

For $\sqrt[3]{\frac{p}{q}}=\left(\frac{q+p-q}{q}\right)^{\frac{1}{3}}=\left(1+\frac{p-q}{q}\right)^{\frac{1}{3}}.$

Now since p is nearly equal to q, $p-q$ is *very small.* Therefore neglecting highers powers of $\frac{p-q}{q}$ than the first, we have

$$\sqrt[3]{\frac{p}{q}}=1+\frac{p-q}{3q}\text{ nearly,}$$

$$=\frac{p+2q}{3q}\ \ldots$$

$$=\frac{q+2p}{p+2q}\ \ldots\ \textit{for } p \textit{ is nearly equal to } q.$$

$$=\frac{2p+q}{p+2q}\ \ldots$$

(xii) If I be integral part of $(19+6\sqrt{10})^n$, show that I is of the form $20M+1$ or $20M-3$, according as n is even or odd.

Let F be the fractional part in $(19+6\sqrt{10})^n$;

$$\therefore\ I+F=(19+6\sqrt{10})^n=(3+\sqrt{10})^{2n}$$

$$=3^{2n}+\frac{2n}{1}.3^{2n-1}.10^{\frac{1}{2}}+\frac{2n(2n-1)}{1.2}10+\ldots+10^n, \qquad \text{(i)}.$$

Let F' denote $(3-\sqrt{10})^{2n}$.

Then since $(3+\sqrt{10})^{2n}>1$ obviously,

and $(3+\sqrt{10})^{2n}(3-\sqrt{10})^{2n}=(9-10)^{2n}=1$;

$\therefore$ *F' is a proper fraction.*

Now $$F'=3^{2n}-\frac{2n}{1}.3^{2n-1}.10^{\frac{1}{2}}+\frac{2n(2n-1)}{1.2}10-\ldots+10^n, \qquad \text{(ii)}.$$

The last term being $+$, since the number of terms $(2n+1)$ is odd.

Hence adding (i) and (ii), we have

$$I+F+F'=2\{3^{2n}+10p\}, \text{ say,}$$

for all terms except the first contain 10.

Now the right-hand side being integral $F+F'=1$;

$$\therefore\ I+1=2\{(10-1)^n+10p\};$$

$$=2\left\{10^n-\frac{n}{1}.10^{n-1}+\ldots+(-1)^n+10p\right\};$$

$\therefore$ $I+1$ is of the form $2\{10M+(-1)^n\}$;

$\therefore$ I is of the form $20M+2(-1)^n-1$,

$=20M+1$ if n be even,

or $=20M-3$ if n be odd.

MULTINOMIAL THEOREM.

113. To find the term involving $a^p b^q c^r \ldots$ in the expansion of $(a+b+c+\ \ldots)^m$.

Let $(b+c+\ \ldots)=y$; $\therefore (a+b+c+\ \ldots)^m=(a+y)^m$.

Let $p=m-s$; $\therefore$ the term involving a^p is

$$\frac{m(m-1)\ \ldots\ (p+1)}{\underline{|s}}a^p y^s,$$

where s is *always* a positive integer, but where the value of p depends on that of m. If m be a positive integer, so also is p, but not otherwise.

(i) Let m be a positive integer.

Then the term involving a^p may be written

$$\underline{|m}\,\frac{a^p}{\underline{|p}}\cdot\frac{y^s}{\underline{|s}}.$$

Now $y^s=(b+c+\ \ldots)^s=(z+b)^s$ say, and the term involving b^q in this is

$$\frac{s(s-1)\ \ldots\ (s-q+1)}{\underline{|q}}z^{s-q}b^q,$$

q being always a positive integer.

Let $s-q=\alpha$ and this becomes

$$\frac{\underline{|s}}{\underline{|q}\,\underline{|\alpha}}z^\alpha b^q;$$

$\therefore$ the term involving $a^p b^q$ is $\underline{|m}.\dfrac{a^p}{\underline{|p}}\ \dfrac{b^q}{\underline{|q}}\ \dfrac{z^\alpha}{\underline{|\alpha}}.$

Similarly the term involving $a^p b^q c^r \ldots$ is

$$\underline{|m}.\frac{a^p}{\underline{|p}}\ \frac{b^q}{\underline{|q}}\ \frac{c^r}{\underline{|r}}\ \ldots$$

where $m=p+s=p+q+\alpha=p+q+r+\ \ldots$

(ii) Let m be *not* a positive integer.

Then all the reasoning above holds, except that in which we write

$$\frac{m.(m-1)\ldots(p+1)}{\lfloor\underline{s}}.=\frac{\lfloor\underline{m}}{\lfloor\underline{s}\,\lfloor\underline{p}},$$

and the term involving $a^pb^qc^r\ldots$ is therefore

$$\{m.(m-1)\ldots(p+1)\}.a^p\frac{b^q}{\lfloor\underline{q}}\ \frac{c^r}{\lfloor\underline{r}}\ \ldots$$

where p is not a positive integer, but $q, r, \ldots$ are so restricted.

From the preceding work it appears that

$$(a+b+c+d+\ldots)^m=\Sigma\lfloor\underline{m}.\frac{a^p}{\lfloor\underline{p}}\ \frac{b^q}{\lfloor\underline{q}}\ \frac{c^r}{\lfloor\underline{r}}\ \ldots$$

where Σ denotes "the sum of such things as."

114. *To find the number of terms in the expansion of $(a+b+c+\ldots$ to n terms$)^m$.*

This is the same obviously as the number of homogeneous products of m dimensions, which can be formed of n quantities and their powers.

Therefore the number of terms

$$=\frac{n(n+1)\ldots(n+m-1)}{\lfloor\underline{m}},\quad .\quad \text{(Art. 108).}$$

115. Since $(a+b+c+\ldots)^m=\Sigma\lfloor\underline{m}.\dfrac{a^p}{\lfloor\underline{p}}\ \dfrac{b^q}{\lfloor\underline{q}}\ \dfrac{c^r}{\lfloor\underline{r}}\ldots$

$$\therefore\ (a+bx+cx^2+\ldots)^m=\Sigma\lfloor\underline{m}.\frac{a^p}{\lfloor\underline{p}}\ \frac{(bx)^q}{\lfloor\underline{q}}\ \frac{(cx^2)^r}{\lfloor\underline{r}}\ldots$$

$$=\Sigma\lfloor\underline{m}.\frac{a^pb^qc^r}{\lfloor\underline{p}\,\lfloor\underline{q}\,\lfloor\underline{r}\ldots}x^{q+2r+\ldots}$$

Hence we can pick out the term involving any power of x in

the expansion (x^n say), for we have merely to make $p, q, r, \ldots$ satisfy the conditions

$$q+2r+\ldots=n,$$
$$p+q+r+\ldots=m.$$

116. *Examples.*

(i) Find the term involving x^4 in $(1+2x+3x^2)^5$.

Here the general term $=\underline{|5}\cdot\frac{1}{\underline{|p}}\cdot\frac{2^q}{\underline{|q}}\cdot\frac{3^r}{\underline{|r}}x^{q+2r}$;

$$\left.\begin{array}{r}\therefore\ q+2r=4\\ \text{and }p+q+r=5\end{array}\right\}\quad\begin{array}{r}\text{(i),}\\ \text{(ii).}\end{array}$$

Now the values of p, q, r, which satisfy these two equations are to be found in the following table :—

p	q	r
3	0	2
2	2	1
1	4	0

Therefore the term required is

$$\underline{|5}\left\{\frac{1}{\underline{|3}}\cdot\frac{3^2}{\underline{|2}}+\frac{1}{\underline{|2}}\cdot\frac{2^2}{\underline{|2}}\cdot\frac{3}{\underline{|1}}+\frac{1}{\underline{|1}}\cdot\frac{2^4}{\underline{|4}}\cdot\right\}x^4$$
$$=(90+120+80)x^4=290x^4.$$

Note.—In drawing up this table, consider from (i) what is the highest admissible value for r. Take q such that with this value (i) is satisfied, and finally take that value of p, which, *with the values of q and r already written down*, satisfies the equation (ii).

Next make r one less, and proceed as before, and repeat the process as often as the equations admit of it.

(ii) Find the term involving x^3 in $(a+bx+cx^2)^{\frac{3}{2}}$.

The general term is $\underline{\left|\frac{3}{2}\right.}\cdot\frac{a^p}{\underline{|p}}\cdot\frac{b^q}{\underline{|q}}\cdot\frac{c^r}{\underline{|r}}\cdot x^{q+2r}$.

Here then $\left.\begin{array}{r} q+2r=3 \\ p+q+r=\frac{3}{2} \end{array}\right\}.$

p	q	r
$-\frac{1}{2}$	1	1
$-\frac{3}{2}$	3	0

Hence the required term is

$$\underline{|\tfrac{3}{2}}\left\{\frac{a^{-\frac{1}{2}}bc}{\underline{|-\frac{1}{2}}}+\frac{a^{-\frac{3}{2}}b^3}{\underline{|-\frac{3}{2}}\,\underline{|3}}\right\}x^3$$

$$=\left\{\frac{3}{2}\cdot\frac{1}{2}\underline{|-\tfrac{1}{2}}\cdot\frac{a^{-\frac{1}{2}}bc}{\underline{|-\frac{1}{2}}}+\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\underline{|-\tfrac{3}{2}}\cdot\frac{a^{-\frac{3}{2}}b^3}{\underline{|-\frac{3}{2}}\,\underline{|3}}\right\}x^3,$$

$$=\left\{\frac{3bc}{4a^{\frac{1}{2}}}-\frac{b^3}{16a^{\frac{3}{2}}}\right\}x^3.$$

The last step but one should be carefully noticed.

(iii) Find the coefficient of x^3 in $(a+bx+cx^2)^{-2}$.

The general term is $\underline{|-2}\,\dfrac{a^p}{\underline{|p}}\,\dfrac{b^q}{\underline{|q}}\,\dfrac{c^r}{\underline{|r}}\,x^{q+2r}$.

Here then $\begin{array}{r} q+2r=\ \ 3, \\ p+q+r=-2. \end{array}$

p	q	r
-4	1	1
-5	3	0

Hence the term required is

$$\underline{|-2}\left\{\frac{a^{-4}bc}{\underline{|-4}}+\frac{a^{-5}b^3}{\underline{|-5}\,\underline{|3}}\right\}x^3,$$

$$=\left\{(-2)(-3)\underline{|-4}.\frac{bc}{a^4\underline{|-4}}+(-2)(-3)(-4)\underline{|-5}\frac{b^3}{\underline{|-5}\,.\,\underline{|3}\,a^5}\right\}x^3,$$

$$=\left\{\frac{6bc}{a^4}-\frac{4b^3}{a^5}\right\}x^3.$$

INDETERMINATE COEFFICIENTS.

117. If $A+Bx+Cx^2+\ldots=a+bx+cx^2+\ldots$ *identically*, then $A=a$, $B=b$, $C=c\ldots$

(i) Let $A+Bx=a+bx$;

$$\therefore\ (A-a)+(B-b)x=0.$$

Now unless $A-a=0$ and $B-b=0$, this is a simple equation which cannot be satisfied by more than one value of x. But by Hypothesis it *is* satisfied by *any* value of x;

$$\therefore\ A-a=0 \text{ and } B-b=0,$$
$$\text{whence } A=a \text{ and } B=b.$$

(ii) Let $A+Bx+Cx^2=a+bx+cx^2$;

$$\therefore\ (A-a)+(B-b)x+(C-x)x^2=0.$$

Now unless $A-a=0$, $B-b=0$, and $C-c=0$, this is a quadratic which cannot be satisfied by more than two values of x; but this is contrary to the Hypothesis;

$$\therefore\ A-a=0,\quad B-b=0,\quad \text{and}\quad C-c=0;$$
$$\therefore\ A=a,\quad B=b,\quad \text{and}\quad C=c.$$

Similarly, the theorem may be demonstrated whatever be the number of terms taken.

118. It may be thought that this reasoning fails when the number of terms taken on each side is *infinitely great.* For the coefficients $(A-a)$, $(B-b)\ldots$ have been equated to zero on the ground that by Hypothesis x has more values than the degree of the equation warrants. Now if the degree of the equation be *infinite*, how, it may be asked, can there be more values of x than the infinite number of roots given by such an equation. The answer is very simple. These roots are essentially *discontinuous*, though infinite in number: the values which x may receive according *to the hypothesis* are also infinite in number, but *continuous.* The infinitude in the latter case is

therefore much greater than in the former, and therefore the coefficients $(A-a)$, $(B-b)$ . . . must be severally zero as before.

As an example of what we mean by "continuous," and "discontinuous," we may remark that a piano is a discontinuous, a violin a continuous, instrument: between any two notes on the latter as the finger is passed along the string there are infinite gradations, between two notes on the former there is nothing of the kind.

119. *Example.*

Find $1^2+2^2+3^2+\ .\ .\ .\ +n^2$.

Let $1^2+2^2+\ .\ .\ .\ +n^2=a+bn+cn^2+dn^3+en^4\ .\ .\ .$

$$\therefore\ 1^2+2^2+3^2+\ldots+(n+1)^2=a+b(n+1)+c(n+1)^2$$
$$+d(n+1)^3+e(n+1)^4+\ldots$$

$\therefore\ n^2+2n+1=3dn^2+(2c+3d)n+(b+c+d)n$, by subtraction, for it is plain that ef . . . are severally 0, as there is no term involving n^3n^4 . . . on the left-hand side.

Hence $3d=1$, $2c+3d=2$, and $b+c+d=1$;

$\therefore\ d=\frac{1}{3}=c=\frac{1}{2}$, and $b=\frac{1}{6}$. Also by putting $n=0$ in the original assumption we obtain $a=0$;

$$\therefore\ 1^2+2^2+\ldots+n^2=\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}=\frac{n\ 1+3n+2n^2)}{6},$$

$$=\frac{n(n+1)(2n+1)}{6},\ \text{as in Art. 67, Cor.}$$

An examination of this example shows that in our series we need not have written down any term not involving n, for by putting $n=0$ it is seen that this term vanishes.

Moreover, we need not have taken terms beyond dn^3, so for $1^3+2^3+\ .\ .\ .\ +n^3$ we need not proceed beyond en^4, and generally for $1^r+2^r+\ .\ .\ .\ +n^r$ we need not proceed beyond tn^{r+1}.

By exactly the same method as above we can prove that

$$1^3+2^3+\ldots+n^3=\left\{\frac{n(n+1)}{2}\right\}^2=(1+2+3+\ldots+n)^2.$$

Both of these examples are however merely particular cases of the more general problem which we discuss in the next section.

120. To find $(1^r+2^r+3^r+\ldots+n^r)$.

Let $1^r+2^r+\ldots+n^r=a_0n^{r+1}+a_1n^r+a_2n^{r-1}+\ldots+a_rn$;

$$\therefore\ 1^r+2^r+\ldots+(n+1)^r$$
$$=a_0(n+1)^{r+1}+a_1(n+1)^r+a_2(n+1)^{r-1}+\ldots+a_r(n+1);$$

$\therefore$ by subtraction

$$n^r+\frac{r}{1}.n^{r-1}+\ldots+\frac{r(r-1)\ldots(r-s+1)}{\underline{|s}}n^{r-s}+\ldots$$
$$=a_0(r+1)n^r+\left\{a_0\frac{(r+1)r}{1.2}+a_1\frac{r}{1}\right\}n^{r-1}+\ldots$$
$$+\left\{a_0\frac{(r+1)r\ldots(r-s+1)}{\underline{|s+1}}+a_1\frac{r\ldots(r-s+1)}{\underline{|s}}\right.$$
$$\left.+a_2\frac{(r-1)\ldots(r-s+1)}{\underline{|s-1}}+\ldots+a_s\frac{r-s+1}{1}\right\}n^{r-s}+\ldots$$

Hence equating coefficients we have $a_0=\frac{1}{r+1}$, $a_1=\frac{1}{1.2}$, and

$$\frac{\underline{|r}}{\underline{|s+1}\,\underline{|r-s}}+\frac{\underline{|r}}{2\underline{|s}\,\underline{|r-s}}+a_2\frac{\underline{|r-1}}{\underline{|s-1}\,\underline{|r-s}}+\ldots$$
$$+a_s\frac{\underline{|r-s+1}}{\underline{|1}\,\underline{|r-s}}=\frac{\underline{|r}}{\underline{|s}\,\underline{|r-s}}.$$

$\therefore$ Multiplying by $\underline{|r-s}$ and bringing all the terms to the left-hand side we obtain

$$\frac{\underline{|r}}{\underline{|s+1}}-\frac{\underline{|r}}{2\underline{|s}}+\frac{a_2\underline{|r-1}}{\underline{|s-1}}+\frac{a_3\underline{|r-2}}{\underline{|s-2}}+\ldots\frac{a_s\underline{|r-s+1}}{\underline{|1}}=0.$$

In this put $s=2, 3, 4, \ldots$ successively, and we obtain

$$\frac{r}{\underline{|3}}-\frac{r}{2\underline{|2}}+a_2=0, \qquad \text{whence } a_2=\frac{r}{3.4},$$

$$\frac{r(r-1)}{\underline{|4}}-\frac{r(r-1)}{2\underline{|3}}+\frac{r.(r-1)}{3.4\underline{|2}}+a_3=0, \text{ whence } a_3=0.$$

Similarly we may show that $a_4=-\frac{1}{30}$, $a_5=0$, and so on for each of the coefficients.

Cor. Since $1^r+2^r+\ldots+n^r=n^{r+1}\left\{\frac{1}{r+1}+\frac{a_1}{n}+\frac{a_2}{n^2}+\ldots\right\}$;

$\therefore$ r being finite, and so also $a_1 a_2 \ldots$ we have the *limit of*

$$\frac{1^r+2^r+\ldots+n^r}{n^{r+1}}=\frac{1}{r+1} \text{ when } n \text{ is made infinite.}$$

121. The following is another typical example of the use of indeterminate coefficients.

Given $y=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$ express x in terms of y.

In this if we write $-x$ for x we obtain $-y$, hence we must assume for x such a series that if we write $-y$ for y we obtain $-x$. That is, we must assume

$$x=ay+by^3+cy^5+\ldots .$$

$$\therefore\ -\frac{x^3}{3}=-\frac{a^3y^3}{3}-a^2by^5-\ldots$$

$$\frac{x^5}{5}= \qquad \frac{a^5}{5}+\ldots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Therefore by addition we have

$$y=ay+\left(b-\frac{a^3}{3}\right)y^3+\left(c-a^2b+\frac{a^5}{5}\right)y^5+\ldots$$

whence equating coefficients we have

$$\therefore\ a=1, \quad b=\tfrac{1}{3}, \quad c=\tfrac{2}{15}\ldots$$

$$\therefore\ x=y+\frac{y^3}{3}+\frac{2y^5}{15}+\ldots$$

This process is sometimes called "reverting the series."

PARTIAL FRACTIONS.

122. Let $\dfrac{f(x)}{F(x)}$ be a fraction, the numerator of which is of lower dimensions than the denominator.

Let $F(x)=(x-a)(x-b)^p(x^2+cx+d)(x^2+cx+f)^q$. Whence if $F(x)$ be of n dimensions, we have

$$1+p+2+2q=n.$$

Then $\dfrac{f(x)}{F(x)}$ may be expressed as the sum or difference of a number of fractions called its Partial Fractions, by the method of Indeterminate Coefficients.

The assumptions to be made are as follows :—

For every factor of the first degree, say $(x-a)$ assume a fraction $\dfrac{A}{x-a}$.

For every factor of the first degree *repeated*, say $(x-b)^r$ assume a series of fractions

$$\frac{B_1}{(x-b)}+\frac{B_2}{(x-b)^2}+\ \dots\ +\frac{B_r}{(x-b)^r}.$$

For every factor of the second degree, say (x^2+cx+d) assume a fraction

$$\frac{Cx+D}{x^2+cx+d},$$

and for every factor of the second degree *repeated*, say $(x^2+ex+f)^q$ assume a series of fractions,

$$\frac{E_1x+F_1}{x^2+ex+f}+\frac{E_2x+F_2}{(x^2+ex+f)^2}+\ \dots\ +\frac{E_qx+F_q}{(x^2+ex+f)^q}.$$

The reasons for these assumptions the student will easily see for himself.

We subjoin a few examples to show the method of working, which is always tolerably simple.

(i) Separate $\dfrac{1}{(x-a)(x-b)}$ into partial fractions.

Let $\dfrac{1}{(x-a)(x-b)}=\dfrac{A}{x-a}+\dfrac{B}{x-b}$;

$$\therefore\ 1=A(x-b)+B(x-a).$$

Put $x=a$ and we have $A=\dfrac{1}{a-b}$;

„ $x=b$ „ $B=-\dfrac{1}{a-b}$.

Hence $\dfrac{1}{(x-a)(x-b)}=\dfrac{1}{(a-b)(x-a)}-\dfrac{1}{(a-b)(x-b)}$.

(ii) Separate $\dfrac{x^2-x+1}{x^2(x+1)}$ into partial fractions.

Let $\dfrac{x^2-x+1}{x^2(x+1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+1}$;

$$\therefore\ \frac{x^2-x+1}{x^2}=C+(x+1)\left\{\frac{A}{x}+\frac{B}{x^2}\right\},$$

put $x=-1$; $\therefore\ C=3$;

$$\therefore\ \frac{A}{x}+\frac{B}{x^2}=\frac{x^2-x+1}{x^2(x+1)}-\frac{3}{x+1},$$

$$=\frac{1-x-2x^2}{x^2(x+1)},$$

$$=\frac{1-2x}{x^2};$$

$$\therefore\ Ax+B=1-2x,$$

whence $B=1$ and $A=-2$ by equating coefficients;

$$\therefore\ \frac{x^2-x+1}{x^2(x+1)}=\frac{3}{x+1}-\frac{2}{x}+\frac{1}{x^2}.$$

(iii) Separate $\frac{1}{x^3+a^3}$ into partial fractions.

Let $\frac{1}{(x+a)(x^2-ax+a^2)}=\frac{A}{x+a}+\frac{Bx+C}{x^2-ax+a^2}$;

$$\therefore \frac{1}{x^2-ax+a^2}=A+(x+a)\{\text{etc.}\}.$$

Put $x=-a$; $\therefore A=\frac{1}{3a^2}$;

$$\therefore \frac{Bx+C}{x^2-ax+a^2}=\frac{1}{x^3+a^3}-\frac{1}{3a^2(x+a)}=\frac{3a^2-x^2+ax-a^2}{3a^2(x^3+a^3)}$$

$$=\frac{2a^2+ax-x^2}{3a^2(x^3+a^3)}=\frac{(2a-x)(a+x)}{3a^2(x^3+a^3)}=\frac{2a-x}{3a^2(x^2-ax+a^2)};$$

$$\therefore Bx+C=\frac{2a-x}{3a^2},$$

whence by equating coefficients we obtain

$$B=-\frac{1}{3a^2} \text{ and } C=\frac{2a}{3a^2}.$$

Thus $\frac{1}{x^3+a^3}=\frac{1}{3a^2(x+a)}-\frac{x-2a}{3a^2(x^2-ax+a^2)}$.

Aliter.—B and C might have been found *first*, however, thus:—

Since $\frac{1}{(x+a)(x^2-ax+a^2)}=\frac{Bx+C}{x^2-ax+a^2}+\frac{A}{x+a}$;

$$\therefore \frac{1}{x+a}=(Bx+C)+(x^2-ax+a^2)(\text{etc.}).$$

Put $x^2=ax-a^2$, and we have

$$1=(Bx+C)(x+a)=Bx^2+Cx+aBx+aC,$$
$$=B(ax-a^2)+Cx+aBx+aC,$$
$$=(C+2aB)x+(aC-a^2B).$$

This is true whenever $x^2=ax-a^2$, that is for *two* different

values of x, although it appears to be a simple equation, $\therefore$ it is really an identity, whence by equating coefficients

$$C+2aB=0 \text{ and } aC-a^2B=1,$$

and from these we obtain

$$B=-\frac{1}{3a^2} \text{ and } C=\frac{2a}{3a^2} \text{ as before.}$$

(iv) Separate $\dfrac{x+2}{(x^2+1)(x-1)^3}$ into partial fractions.

Let

$$\frac{x+2}{(x^2+1)(x-1)^3}=\frac{Ax+B}{x^2+1}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^2}+\frac{E}{x-1};$$

$$\therefore\ x+2=(Ax+B)(x-1)^3+$$
$$\{C+D(x-1)+E(x-1)^2\}(x^2+1), \qquad \text{(i).}$$

Let $x-1=h$; $\therefore\ x=1+h$ and we have

$$3+h=(Ax+B)h^3+(C+Dh+Eh^2)(2+2h+h^2).$$

Now equate coefficients of the same powers of h, and

$$2C=3; \qquad \therefore\ C=\tfrac{3}{2},$$
$$2C+2D=1; \qquad \therefore\ D=-1,$$
$$C+2D+2E=0; \qquad \therefore\ E=\tfrac{1}{4}.$$

In (i) put $x^2=-1$, and

$$x+2=(Ax+B)(-x+3+3x-1)=(Ax+B)(2x+2);$$
$$\therefore\ -2A+2B=2, \text{ and } 2A+2B=1,$$
$$\text{whence } B=\tfrac{3}{4} \text{ and } A=-\tfrac{1}{4};$$

$\therefore$ the given fraction

$$=\frac{-x+3}{4(x^2+1)}+\frac{3}{2(x-1)^3}-\frac{1}{(x-1)^2}+\frac{1}{4(x-1)}.$$

(v) *Find the coefficient of x^n in the expansion of*

$$\frac{1}{x^2+2px+q}.$$

Let hk be the roots of $x^2+2px+q=0$, and assume

$$\frac{1}{x^2+2px+q}=\frac{A}{(x-h)}+\frac{B}{x-k};$$

$$\therefore\ 1=A(x-k)+B(x-h), \text{ whence}$$

$$A=\frac{1}{h-k} \text{ and } B=-\frac{1}{h-k};$$

$$\therefore \text{ the given fraction } =\frac{1}{h-k}\,\frac{1}{x-h}-\frac{1}{h-k}\,\frac{1}{x-k}$$

$$=-\frac{1}{h-k}\left\{\frac{1}{h\left(1-\frac{x}{h}\right)}-\frac{1}{k\left(1-\frac{x}{k}\right)}\right\}$$

$$=\frac{1}{h-k}\left\{\frac{1}{k}\left(1-\frac{x}{k}\right)^{-1}-\frac{1}{h}\left(1-\frac{x}{h}\right)^{-1}\right\};$$

$\therefore$ the coefficient of x^n required is

$$\frac{1}{(h-k)k}\,\frac{1}{k^n}-\frac{1}{(h-k)h}\cdot\frac{1}{h^n}=\frac{h^{n+1}-k^{n+1}}{(h-k)(hk)^{n+1}}=\frac{h^{n+1}-k^{n+1}}{(h-k)q^{n+1}},$$

for $hk=q$.

(vi) *To find the coefficient of x^n in the expansion of* $(1+x^3)^{-1}$.

Let h, k, be the roots of $x^2-x+1=0$, so that $h+k=1$ and $hk=1$;

$$\therefore\ 1-x+x^2=1-(h+k)x+hkx^2=(1-hx)(1-kx).$$

Assume therefore

$$\frac{1}{1+x^3}=\frac{A}{1+x}+\frac{B}{1-hx}+\frac{C}{1-kx};$$

$$\therefore\ 1=A(1-hx)(1-kx)+B(1+x)(1-kx)+C(1+x)(1-hx).$$

Put $x=-1$, and $1=A(1+h)(1+k)=A(1+\overline{h+k}+hk)$,

$=A(1+1+1)$; $\quad\therefore\ A=-\frac{1}{3}$.

Put $x=\frac{1}{h}$ and $1=B\left(1+\frac{1}{h}\right)\left(1-\frac{k}{h}\right)=\frac{B(1+k)(h-k)}{h}$

for $\frac{1}{h}=k$; $\quad\therefore\ B=\frac{h}{(1+k)(h-k)}$; so $C=-\frac{k}{(1+h)(h-k)}$.

Now coefficient of x^n required $=A(-1)^n+Bh^n+Ck^n$,

$$=\frac{1}{3}(-1)^n+\frac{1}{h-k}\left(\frac{h^{n+1}}{1+k}-\frac{k^{n+1}}{1+h}\right).$$

CONTINUED FRACTIONS.

123. Suppose $\frac{a}{b}$ an improper fraction, and let b be contained in a, p times, with remainder c (c being $\therefore$ necessarily less than b).

Then $\frac{a}{b}=p+\frac{c}{b}=p+\cfrac{1}{\frac{b}{c}}\cdot$

Now let c be contained in b, q times, with remainder d;

$$\therefore\ \frac{a}{b}=p+\cfrac{1}{q+\frac{d}{c}}=p+\cfrac{1}{q+\cfrac{1}{\frac{c}{d}}}\cdot$$

Again, let d be contained in c, r times, with remainder e, and we have

$$\frac{a}{b}=p+\cfrac{1}{q+\cfrac{1}{r+\frac{e}{d}}}\cdot$$

This latter fraction is called a *continued fraction*, and for the sake of convenience is expressed thus:

$$p+\frac{1}{q+}\,\frac{1}{r+}\,\frac{e}{d}, \qquad . \quad . \quad . \qquad \text{(i).}$$

The more *general* type of a continued fraction is however $p\pm\frac{q}{r\pm}\,\frac{t}{s\pm}$ . . . but as we shall deal only with the form (i), we prefer to make that form the subject of our fundamental investigation.

Now from the subjoined scheme it will be seen that the process by which we obtain $p, q, r, \ldots$ which are called **the Partial Quotients** is simply the method of G.C.M.

$$\begin{array}{l} b\,)\,a\,(\,p \\ \quad \underline{pb} \\ \quad\; c\,)\,b\,(\,q \\ \qquad\;\; \underline{qc} \\ \qquad\;\; d\,)\,c\,(\,r \\ \qquad\qquad \underline{rd} \\ \qquad\qquad e\,)\,d\,(\,. \end{array}$$

Consequently any fraction $\frac{a}{b}$ can be turned into a continued fraction which will *terminate*, for the method of G.C.M. can always be applied until, under the most unfavourable circumstances, we have a remainder 1, this being taken as a divisor we obtain a final quotient and no remainder.

Example. Express $\frac{233}{154}$ as a continued fraction.

$$\begin{array}{l} 154\,)\,233\,(\,1 \\ \qquad\;\; \underline{154} \\ \qquad\quad 79\,)\,154\,(\,1 \\ \qquad\qquad\quad\;\; \underline{79} \\ \qquad\qquad\quad\;\; 75\,)\,79\,(\,1 \\ \qquad\qquad\qquad\quad\;\; \underline{75} \\ \qquad\qquad\qquad\qquad 4\,)\,75\,(\,18 \\ \qquad\qquad\qquad\qquad\quad\;\; \underline{72} \\ \qquad\qquad\qquad\qquad\quad\;\; 3\,)\,4\,(\,1 \\ \qquad\qquad\qquad\qquad\qquad\quad \underline{3} \\ \qquad\qquad\qquad\qquad\qquad\quad 1\,)\,3\,(\,3 \\ \qquad\qquad\qquad\qquad\qquad\qquad\;\; \underline{3} \\ \qquad\qquad\qquad\qquad\qquad\qquad\;\; . \end{array}$$

$$\therefore \frac{233}{154}=1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{18+}\,\frac{1}{1+}\,\frac{1}{3}.$$

Note.—If $\frac{a}{b}$ be a *proper* fraction we write

$$\frac{a}{b}=\frac{1}{\frac{b}{a}}=\frac{1}{p+}\,\frac{1}{q+}\,\ldots$$

124. By expressing in the form of a continued fraction a fraction whose numerator and denominator are large, we may obtain approximations to its value.

Thus since $\frac{3471}{1261}=2+\frac{1}{1+}\,\frac{1}{2+}\,\ldots$

we have for successively approximative values

$$2,\ 3,\ 2\tfrac{2}{3}\ \ldots$$

125. *The approximations to the value of* $\frac{a}{b}$ *(an improper fraction) found by taking* 1, 2, 3 . . . *partial quotients, are alternately less and greater than the true value.*

For since $\frac{a}{b}=p+\frac{1}{q+}\,\frac{1}{r+}\,\ldots$

p the first approximation is clearly too little.

$p+\frac{1}{q}$ the second approximation is too great for the denominator q is too small.

$p+\cfrac{1}{q+\cfrac{1}{r}}$ the third approximation is too small, for we reject quantities following r, and thus take $\frac{1}{r}$ instead of a smaller quantity; that is $q+\frac{1}{r}$ instead of a smaller quantity; that is $\cfrac{1}{q+\cfrac{1}{r}}$ instead of a *greater* quantity; that is $p+\frac{1}{q+}\,\frac{1}{r}$ instead of a greater quantity: in other words $p+\frac{1}{q+}\,\frac{1}{r+}\,\ldots$ is less than the true value.

Similarly we may reason for the fourth approximation.

Note.—The approximations p, $p+\frac{1}{q}$, $p+\frac{1}{q+}\,\frac{1}{r+}\,\ldots$ are called the first, second, third . . . **CONVERGENTS**,

and from the above it appears that if $\frac{a}{b}$ be an improper fraction, then the odd convergents are less and the even convergents greater than the true value.

A similar consideration shows that if $\frac{a}{b}$ be a proper fraction then the odd convergents are greater, and the even convergents less, than the true value.

125. We denote the first, second, third . . . convergents by $\frac{N_1}{D_1}, \frac{N_2}{D_2}, \frac{N_3}{D_3}$ . . . and the corresponding partial quotients by q_1, q_2, q_3 . . . when however the *number* of the convergent is not specified, and we are dealing with *any* convergent, we denote it and those immediately following it by $\frac{n}{d}, \frac{n'}{d'}, \frac{n''}{d''}$, . . . and the corresponding partial quotients by q, q', q'', . . .

Hence $\frac{N_1}{D_1}=\frac{q_1}{1}$. $\qquad \frac{N_2}{D_2}=\frac{q_1q_2+1}{q_2}$

$$\frac{N_3}{D_3}=q_1+\frac{1}{q_2+}\,\frac{1}{q_3} \qquad =\frac{q_3(q_1q_2+1)+q_1}{q_3q_2+1} \text{ on reduction.}$$

$$\frac{N^4}{D_4}=q_1+\frac{1}{q_2+}\,\frac{1}{q_3+}\,\frac{1}{q_4}=\frac{q_4\{q_3(q_1q_2+1)+q_1\}+q_1q_2+1}{q_4(q_3q_2+_1)+q_2}.$$

Hence we may *conjecture* the following laws,

$$N_n=q_nN_{n-1}+N_{n-2}$$
$$D_n=q_nD_{n-1}+D_{n-2}$$

as the laws of formation for the convergents.

126. To prove $\frac{N_n}{D_n}=\frac{q_nN_{n-1}+N_{n-2}}{q_nD_{n-1}+D_{n-2}}$

assume $N_{n-1}=q_{n-1}N_{n-2}+N_{n-3}$,

and $D_{n-1}=q_{n-1}D_{n-2}+D_{n-3}$.

Now $\frac{N_n}{D_n}$ differs from $\frac{N_{n-1}}{D_{n-1}}$ only in that we write $q_{n-1}+\frac{1}{q_n}$ instead of q_{n-1}.

Hence $\dfrac{N_n}{D_n}=\dfrac{\left(q_{n-1}+\dfrac{1}{q_n}\right)N_{n-2}+N_{n-3},}{\left(q_{n-1}+\dfrac{1}{q_n}\right)D_{n-2}+D_{n-3},}$

$$=\frac{q_{n-1}q_nN_{n-2}+N_{n-2}+q_nN_{n-3}}{q_{n-1}q_nD_{n-2}+D_{n-2}+q_nD_{n-3}}$$

$$=\frac{q_n(q_{n-1}N_{n-2}+N_{n-3})+N_{n-2}}{q_n(q_{n-1}D_{n-2}+D_{n-3})+D_{n-2}}$$

$$=\frac{q_nN_{n-1}+N_{n-2}}{q_nD_{n-1}+D_{n-2}} \text{ from our assumption.}$$

Hence if the assumed law holds for the formation of the $(n-1)$th convergent so also does it hold for the nth. But it *does* hold for the third, therefore for the fourth, therefore for the fifth, and so by mathematical induction universally.

127. **The difference between any two consecutive convergents is a fraction, whose Numerator=1.**

Since the first two convergents are respectively $\frac{q_1}{1}$ and $\frac{q_1q_2+1}{q_2}$;

$\therefore$ the numerator of (the second − the first) $=q_1q_2+1-q_1q_2=1$.

Now assume that this holds for *any* two consecutive convergents $\frac{n}{d}$, $\frac{n'}{d'}$. That is, let the numerator in $\frac{n'}{d'}-\frac{n}{d}$,

viz., $n'd-nd'=\pm 1$.

Then $\dfrac{n''}{d''}-\dfrac{n'}{d'}=\dfrac{n''d'-n'd''}{d'd''}=\dfrac{(q''n'+n)d'-(q''d'+d)n'}{d'd''}$,

$$=\frac{nd'-n'd}{d'd''}=\mp\frac{1}{d'd''};$$

$\therefore$ if the law hold for any two convergents, it holds also for the latter of these two and that next following it; but it does hold for the first and second, therefore for the second and third, and so by mathematical induction universally.

Note. $\dfrac{N_n}{D_n}-\dfrac{N_{n-1}}{D_{n-1}}$ has $(-1)^n$ for its numerator.

For from the above it appears that the numerator of the
second convergent—first convergent $= +1 = (-1)^2$,
third „ —second „ $= -1 = (-1)^3$,
fourth „ —third „ $= +1 = (-1)^4$,
so nth „ $-(n-1)$th „ $= (-1)^n$.

Cor.—**All convergents are in their lowest terms.**

For if $\frac{n}{d}$ are not in their lowest terms, they have an (integral) common measure which therefore measures $n'd - nd' = \pm 1$, which is absurd. Therefore, etc.

128. **Any convergent $\frac{n'}{d'}$ is nearer to the true value $\frac{a}{b}$ than the preceding convergent $\frac{n}{d}$.**

For $\frac{n''}{d''} = \frac{q''n'+n}{q''d'+d}$.

Now $\frac{a}{b}$ differs from $\frac{n''}{d''}$ only in writing $q'' + \cfrac{1}{q''' + \dots}$ for q''.

This latter fraction is obviously > 1; $\therefore$ denoting it by k we have

$$\frac{a}{b} = \frac{kn'+n}{kd'+d},$$

$$\therefore \frac{n}{d} - \frac{a}{b} = \frac{n}{d} - \frac{kn'+n}{kd'+d} = \frac{nkd'+nd-n'kd-nd}{d(kd'+d)},$$

$$= \frac{k(nd'-n'd)}{d(kd'+d)} = \frac{\pm k}{d(kd'+d)} \quad . \quad . \quad . \quad \text{(i)},$$

$$\text{and } \frac{a}{b} - \frac{n'}{d'} = \frac{kn'+n}{kd'+d} - \frac{n'}{d'} = \frac{kn'd'+nd'-kn'd'-n'd}{d'(kd'+d)},$$

$$= \frac{\pm 1}{d'(kd'+d)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

Now the Numerator in (ii) is $<$ Numerator in (i) for k is > 1,

and the Denominator in (ii) is $>$ the Denominator in (i) for d' is $> d$;

$$\therefore \text{ on both accounts (ii) is} < \text{(i}$$

$$\therefore \frac{a}{b} \text{ is nearer to } \frac{n'}{d'} \text{ than to } \frac{n}{d}.$$

129. **The error arising from writing $\frac{n}{d}$ instead of $\frac{a}{b}$ is less than $\frac{1}{d^2}$.**

For the numerical difference between $\frac{n}{d}$ and $\frac{a}{b}$ is

$$\frac{n}{d} - \frac{a}{b} = \frac{n}{d} - \frac{kn'+n}{kd'+d} = \frac{k}{d(kd'+d)}$$

$$= \frac{1}{d\left(d'+\frac{d}{k}\right)}, \text{ and this is}$$

$$\leq \frac{1}{dd'};$$

$$\text{and } \therefore < \frac{1}{d^2} \textit{ a fortiori} \text{ for } d \text{ is} < d'.$$

Thus $\frac{1}{d^2}$ is the *superior* limit of the error. Similarly we may discuss the *inferior* limit.

130. **Any convergent $\left(\frac{n'}{d'}\right)$ is nearer to the true value than any other fraction $\left(\frac{r}{s}\right)$ which has a smaller denominator than d'.**

(i) Let $\frac{r}{s}$ be one of the preceding convergents.

Then the proposition follows at once (128).

(ii) Let $\frac{r}{s}$ be *not* one of the preceding convergents.

Then it cannot lie between $\frac{n}{d}$ and $\frac{n'}{d'}$ or we should have $\frac{r}{s} \sim \frac{n}{d} < \frac{n'}{d'} \sim \frac{n}{d}$;

$\therefore rd \sim ns < \frac{s}{d'}$,

which, since r, s, d, n, are integers, and $s<d'$ is impossible. Hence

$$\frac{r}{s}, \frac{n}{d}, \frac{a}{b}, \frac{n'}{d'}, \text{ or } \frac{n}{d}, \frac{a}{b}, \frac{n'}{d'}, \frac{r}{s},$$

are in order of magnitude.

In the latter case $\frac{n'}{d'}$ *is* nearer to $\frac{a}{b}$ than $\frac{r}{s}$ is.

In the former $\frac{n}{d}$ and $\therefore$ *a fortiori* (128) $\frac{n'}{d'}$ is nearer to $\frac{a}{b}$ than $\frac{r}{s}$ is.

Hence in *either* case $\frac{n'}{d'}$ is nearer to the true value than $\frac{r}{s}$ is.

131. **A convergent $\left(\frac{n'}{d'}\right)$ immediately preceding a large partial quotient (q'') is much nearer to the true value $\left(\frac{a}{b}\right)$ than the preceding convergent $\left(\frac{n}{d}\right)$ is.**

For $\frac{n'}{d'} \sim \frac{a}{b} = \frac{n'}{d'} \sim \frac{kn'+n}{kd'+d} = \frac{1}{d'(kd'+d)} = A_1$ say,

and $\frac{n}{d} \sim \frac{a}{b} = \frac{n}{d} \sim \frac{kn'+n}{kd'+d} = \frac{k}{d(kd'+d)} = A_2$ say.

Now since $k = q'' + \frac{1}{q'''+ \dots}$ and q'' *is large*; $\therefore$ *k is large* and d is $< d'$.

Hence on both accounts A_1 is *much* less than A_2.

Q.E.D.

132. The four following examples are practically bookwork.

(i) Express $\sqrt{a^2+1}$ as a continued fraction.

$$\sqrt{a^2+1}=a+\sqrt{a^2+1}-a=a+\frac{1}{\sqrt{a^2+1}+a}$$

$$=a+\frac{1}{2a+}\,\frac{1}{\sqrt{a^2+1}+a},$$

$$=a+\frac{1}{2a+}\,\frac{1}{2a+}\,\frac{1}{2a+\ldots}$$

(ii) To find the value of the continued fraction,

$$\frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{p+}\,\frac{1}{q+\cdots}$$

the partial quotients recurring in a fixed order. (TYPE.)

Let $x=$ the required value;

$$\therefore\ x=\frac{1}{p+}\,\frac{1}{q+x}=\frac{q+x}{pq+px+1}.$$

This becomes on reduction

$$px^2+pqx-q=0.$$

A quadratic whence we determine x.

(iii) Convert $\sqrt{11}$ into a continued fraction. (TYPE.)

$\sqrt{11}=3+\sqrt{11}-3$ (Notice, 3 is the *integral part* of $\sqrt{11}$),

$$=3+\frac{2}{\sqrt{11}+3}=3+\frac{1}{\frac{\sqrt{11}+3}{2}},$$

so $\dfrac{\sqrt{11}+3}{2}=3+\dfrac{\sqrt{11}-3}{2}=3+\dfrac{1}{\sqrt{11}+3}$ (3 being the *integral part* of $\dfrac{\sqrt{11}+3}{2}$);

$$\therefore\ \sqrt{11}=3+\frac{1}{3+}\,\frac{1}{\sqrt{11}+3}=3+\frac{1}{3+}\,\frac{1}{6+}\,\frac{1}{3+}\,\frac{1}{6+}\ldots.$$

(iv) Given $6^x=2$. Find x in the form of a continued fraction.

Here it is plain that x lies 0 and 1. Let $x=\dfrac{1}{y}$;

$$\therefore\ 6^{\frac{1}{y}}-2;\ \therefore\ 2^y=6.$$

Hence y lies between 2 and 3. Let $y=2+\frac{1}{z}$;

$$\therefore\ 2^{2+\frac{1}{z}}=6;$$

$$\therefore\ 2^{\frac{1}{z}}=\frac{3}{2};$$

$$\therefore\ \left(\frac{3}{2}\right)^z=2.$$

Hence z lies between 1 and 2. Let $z=1+\frac{1}{p}$,

$$\therefore \left(\frac{3}{2}\right)^{1+\frac{1}{p}}=2.$$

$$\therefore \left(\frac{3}{2}\right)^{\frac{1}{p}}=\frac{4}{3};$$

$$\therefore \left(\frac{4}{3}\right)^{p}=\frac{3}{2}.$$

Hence p lies between 1 and $2=1+\frac{1}{q}$ say, and the process above may be similarly continued.

Therefore $x=\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\ldots$

The subjoined example is instructive.

(v) Prove $\left\{\frac{1}{a-}\,\frac{1}{b-}\,\frac{1}{a-}\,\frac{1}{b-\ldots}\right\}\left\{b-\frac{1}{a-}\,\frac{1}{b=}\,\frac{1}{a-\ldots}\right\}=\frac{b}{a}.$

Let $x=\frac{1}{a-}\,\frac{1}{b-}\,\frac{1}{a-}\,\frac{1}{b-}\,\ldots$

$\therefore\ x=\frac{1}{a-}\,\frac{1}{b-x}=\frac{b-x}{ab-ax-1}$;

$\therefore\ ab-ax^2-x=b-x$;

$\therefore\ ax(b-x)-b$;

$\therefore\ x(b-x)=b$;

$\therefore\ \left\{\frac{1}{a-}\,\frac{1}{b-}\,\frac{1}{a-}\,\frac{1}{b-\ldots}\right\}\left\{b-\frac{1}{a-}\,\frac{1}{b-}\,\frac{1}{a-\ldots}\right\}=\frac{b}{a}.$

INEQUALITIES.

133. Many inequalities are worked by assuming the proposition in question to be true, if certain relations deducible from it are true: *these* to be true if certain other relations deducible from them are true; and so on, until finally we arrive at a statement which is known to be true or known to be false. If this final statement be true, so is the proposition: if this statement be false the proposition is false likewise.

Example. Prove $\dfrac{n^2-n+1}{n^2+n+1}<3$.

$$\frac{n^2-n+1}{n^2+n+1}<3,$$

$$\text{if}\quad n^2-n+1<3n^2+3n+3,$$

$$\text{if}\quad 0<2n^2+4n+2,$$

$$\text{if}\quad 0<(n+1)^2 \quad \textit{which it is},$$

Therefore $\dfrac{n^2-n+1}{n^2+n+1}$ is <3.

134. Again, the solution of many inequalities will be best effected by the adoption of various artifices which can be learnt only by practice. The following examples will be suggestive:—

(i) Prove $abc>(a-b+c)(a+b-c)(b+c-a)$,

$$a^2>a^2-(b-c)^2 \text{ obviously};$$

$$\therefore\ a^2>(a-b+c)(a+b-c),$$

$$\text{so } b^2>(b-c+a)(b+c-a),$$

$$\text{and } c^2>(c-a+b)(c+a-b);$$

$$\therefore\ a^2b^2c^2>(a-b+c)^2(a+b-c)^2(b+c-a)^2;$$

$$\therefore\quad abc>(a-b+c)\,(a+b-c)\,(b+c-a).$$

Q.E.D.

(ii) Prove $ab(a+b)+bc(b+c)+ca(c+a)>6abc$.

The left-hand side $=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$.

Now $b^2+c^2>2bc$ obviously;

$$\therefore\ a(b^2+c^2)>2abc,$$
$$\text{so }\ b(c^2+a^2)>2abc,$$
$$\text{and }\ c(a^2+b^2)>2abc;$$

$\therefore$ by addition the required result follows.

(iii) Prove $ab(a+b)+bc(b+c)+ca(c+a)<2(a^3+b^3+c^3)$.

$$\text{Since }\ 0<a^2-2ab+b^2;$$
$$\therefore\ ab<a^2-ab+b^2;$$
$$\therefore\ ab(a+b)<a^3+b^3.$$
$$\text{So }\ bc(b+c)<b^3+c^3,$$
$$\text{and }\ ca(c+a)<c^3+a^3;$$

$\therefore$ by addition the required result follows.

(iv) $a^2+b^2+c^2>ab+bc+ca$.

$$\text{For }\ a^2+b^2>2ab,$$
$$b^2+c^2>2bc,$$
$$\text{and }\ c^2+a^2>2ac;$$

$\therefore$ by addition and dividing both sides by 2, the required result follows.

N.B.—In all of these examples we assume a, b, and c to be unequal.

135. To prove $\left(\dfrac{a+b+\ .\ .\ .\ +k}{n}\right)^n>abc\ .\ .\ .\ k$ **unless the** n **quantities** $a, b, c, \ .\ .\ .\ k$ **are all equal.**

Now $\left(1+\dfrac{a}{nb}\right)^n$ is $>\left\{1+\dfrac{a}{(n-1)b}\right\}^{n-1}$ (Lemma),

for on expansion, the first two terms are the same on each side, after which each term on the left is greater than the corresponding term on the right, and moreover there is one more term on the left than on the right.

$$\text{Now } \left(\frac{a+b+\ .\ .\ .\ +k}{n}\right)^n = a^n\left(\frac{a+b+\ .\ .\ .\ +k}{na}\right)^n$$

$$= a^n\left\{1+\frac{b+c+\ldots+k-(n-1)a}{na}\right\}^n$$

$$> a^n\left\{1+\frac{b+c+\ldots+k-(n-1)a}{(n-1)a}\right\}^{n-1} \quad \text{(Lemma)},$$

$$> a\left\{\frac{b+c+\ .\ .\ .\ +k}{n-1}\right\}^{n-1}$$

$$> ab\left\{\frac{c+\ .\ .\ .\ +k}{n-2}\right\}^{n-2}, \text{ similarly,}$$

$$> abc\ .\ .\ .\ k, \text{ similarly.}$$

Note.—The above important theorem may be written

$$\frac{a+b+\ .\ .\ .\ +k}{n} > \sqrt[n]{abc\ .\ .\ .\ k},$$

and hence it is sometimes enunciated thus:—

"The A.M. between n quantities is greater than their G.M. unless the n quantities are all equal."

136. Examples of the use of the preceding theorem.

(i) Prove $a^3+b^3+c^3 > 3abc$.

Consider the three quantities a^3, b^3, c^3.

$$\text{Then } \frac{a^3+b^3+c^3}{3} > \sqrt[3]{a^3.b^3.c^3},$$

$$> abc;$$

$$\therefore\ a^3+b^3+c^3 > 3abc.$$

(ii) Prove $\dfrac{n(n+1)^2}{4} > \sqrt[n]{(\underline{|n})^3}$.

Consider the n quantities $1^3, 2^3, 3^3, \ .\ .\ .\ n^3$.

$$\text{Then } \frac{1^3+2^3+\ .\ .\ .\ +n^3}{n} > \sqrt[n]{1^3.2^3\ .\ .\ .\ n^3}.$$

$$\therefore\ \frac{n^2(n+1)^2}{4.n} > \sqrt[n]{(\underline{|n})^3};$$

$$\therefore\ \frac{n(n+1)^2}{4} > \sqrt[n]{(\underline{|n})^3}.$$

SCALES OF NOTATION.

137. In the ordinary system of notation the value of the digits increases by powers of ten as we proceed to the left, or, in other words, *decreases* by powers of ten as we proceed to the right.

Thus in 437·29,

The digit 4 denotes 4 groups of $10^2 = 4\times10^2$,

„ 3 „ 3 „ $10 = 4\times10$,

„ 7 „ 7 units $1 = 7\times1$,

and in continuation of the same law,

„ 2 denotes $2\times\frac{1}{10}$,

„ 9 „ $9\times\frac{1}{10^2}$.

The grouping in this system being in tens, the system is called the decimal scale, and ten is called the radix or base (the decimal point is in reality nothing more than a "*full stop*," which serves to indicate which is the units' place).

Now obviously we may form our groups on *any other base*, r say. We must remember, however, that if we do so—

(i) The value of the digits decreases *by powers of* r to the right, and

(ii) The highest digit in the scale will from the nature of the case be $(r-1)$.

138. *To express any number N in any proposed scale.*

Let r be the radix of the proposed scale, and

(i) Let N be *integral.*

Suppose the digits which represent it in the scale of r to be $a_0, a_1, a_2, \ldots$ beginning at the units' place, so that

$$N = a_0 + a_1r + a_2r^2 + \ldots$$

From this it appears that if we divide N by r, the remainder will be a_0 and the digit in the units' place is found.

If we divide the quotient thus obtained, viz.,

$$a_1+a_2r+a_3r^2+\ .\ .\ .$$

by r, the remainder will be a_1; and so the next digit is found. Similarly we may proceed till we have found *all* the digits.

(ii) Let N be *fractional*,
and let the digits which represent it in the scale of r be a_{-1}, a_{-2}, a_{-3}, . . . so that

$$N=a_{-1}r^{-1}+a_{-2}r^{-2}+a_{-3}r^{-3}+\ .\ .\ .$$

From this it appears that if we multiply N by r the *integral* part will be a_{-1}, and thus a_{-1} is found.

Again, if we multiply the *fractional* part of this product, viz.,

$$a_{-2}r^{-1}+a_{-3}r^{-2}+\ .\ .\ .$$

by r, the integral part will be a^{-2}, and so a^{-2} is found. Similarly we may proceed till we have found *all* the digits.

Example. Transform 347 from the decimal into the senary scale.

$$\begin{array}{rrl}
6\,| & 347 & \\
6\,| & 57 & -5 \\
6\,| & 9 & -3 \\
 & 1 & -3
\end{array}$$

Hence the number is expressed in the senary scale by 1335.

139. *The maximum number of n digits in the scale of r is r^n-1 and the minimum, r^{n-1}.*

(i) For the number will be a maximum when each of the n digits is as great as possible, viz., $(r-1)$.

Hence the number

$$\begin{aligned}
&=(r-1)+(r-1)r+(r-1)r^2+\ \dots\ +(r-1)r^{n-1},\\
&=(r-1)(1+r+r^2+\ .\ .\ .\ +r^{n-1}),\\
&=(r-1)\frac{r^n-1}{r-1}=r^n-1.
\end{aligned}$$

(ii) The number is a minimum when the coefficient of r^{n-1} is 1, and each of the other digits 0. That is, the minimum number with n digits is r^{n-1}.

140. *If P contain p digits and Q contain q digits, then PQ contains* $p+q$ or $p+q-1$ *digits.*

For PQ is $< r^p r^q < r^{p+q}$ and so cannot have more than $p+q$ digits, and PQ is *not* $< r^{p-1} r^{q-1}$; $\therefore PQ$ is not $< r^{p+q-2}$, and so cannot have less than $p+q-1$ digits.

$\therefore$ etc.

The same method may be applied to *any* number of factors.

141. *If P contain p digits, and Q contain q digits, then $\frac{P}{Q}$ contains* $(p-q)$ or $(p-q+1)$ *digits.*

For $\frac{P}{Q}$ is $< \frac{r^p}{r^{q-1}}$ for the numerator is too great and the denominator has its minimum value,

$< r^{p-q+1}$,

$<$ least number with $p-q+2$ digits,

and therefore cannot have more than $(p-q+1)$ digits.

Again $\frac{P}{Q} > \frac{r^{p-1}}{r^q}$ for the numerator has its minimum value and the denominator is too great,

$> r^{p-q-1}$,

$>$ least number with $p-q$ digits,

and therefore cannot have less than $(p-q)$ digits.

Therefore, etc.

The proof here given is *independent of the results of Article* 140.

142. If the sum of the digits of any number expressed in the scale of r be divided by $(r-1)$, the remainder is the same as when the number is divided by $(r-1)$.

Let the number be $(a_0+a_1r+a_2r^2+a_3r^3+ \ . \ . \ . \ +a_nr^n)$

$$=(a_0+a_1+\ldots+a_n)+a_1(r-1)+a_2(r^2-1)+\ldots+a_n(r^n-1).$$

Hence $(r-1)$ being contained in every term *except the first*, and this first term being the sum of the digits, the proposition follows.

Cor.—Any number in the *decimal* scale divided by 9, leaves the same remainder as when the sum of its digits is divided by 9.

143. If $PQ=R$ and x, y, z, be the remainders, when P, Q, R respectively are divided by 9, the numbers being expressed in the decimal scale, then will z = the remainder when xy is divided by 9.

Let P, Q, R, $=9\alpha+x$, $9\beta+y$, $9\gamma+z$ respectively.

Then since $PQ=R$ we have

$$81\alpha\beta+9\alpha y+9\beta x+xy=9\gamma+z.$$

Now 9 is contained in every term except the last on each side. Hence z = the remainder when xy is divided by 9.

From this theorem is derived the test of the accuracy of Multiplication called "casting out the nines."

Example.

$$\begin{array}{r}347\\96\\\hline 2082\\3123\\\hline 33312\\\hline\end{array}\qquad\begin{array}{l}\text{Remainder } \left.\begin{array}{l}x=5\\y=6\end{array}\right\}3\\ \\ \\ \\ \quad ,,\quad z=3.\end{array}$$

The test however fails to detect certain

(i) **Errors of Excess.** The *superfluous presence* in the product of digits whose sum is 9, a multiple of 9 or zero.

(ii) **Errors of Defect.** Their *incorrect absence* from the product.

(iii) **Errors of Position.** A wrong *order* of the digits in the product.

We are not therefore justified in saying that when the test *is* satisfied the working has been correctly performed. We *are* however entitled to say, that when the test is *not* satisfied the working is *incorrect*.

PROPERTIES OF NUMBERS.

144. The product of any r consecutive integers is divisible by $\underline{|r}$.

Let n be the least of such integers, and denote the quotient,

$$\frac{n(n+1)\ .\ .\ .\ (n+r-1)}{\underline{|r}} \text{ by } {}^{n}Q_r.$$

We have to prove that ${}^{n}Q_r$ is an integer.

(i) It is clear that ${}^{n}Q_r=\dfrac{n(n+1)}{\underline{|2}}$ is an integer, for of the two factors, n, $(n+1)$, *one must be even.*

(ii) *Assume* the product of any $(r-1)$ consecutive integers to be divisible by $\underline{|r-1}$, in other words, let ${}^{k}Q_{r-1}$ be an integer for all values of k.

Then since ${}^{n}Q_r=\dfrac{n(n+1)\ldots(n+r-1)}{\underline{|r}}$

$$=\frac{n(n+1)\ldots(n+r-2)}{\underline{|r-1}}\left\{1+\frac{n-1}{r}\right\},$$

$$=\frac{(n-1)n\ldots\{(n-1)+r-1\}}{\underline{|r}}+\frac{n(n+1)\ldots(n+r-2)}{\underline{|r-1}},$$

$$={}^{n-1}Q_r+{}^{n}Q_{r-1},$$

and ${}^{n}Q_{r-1}$ is integral by Hypothesis ;

$$\therefore\ {}^{n}Q_r={}^{n-1}Q_r+\text{ an integer,}$$

$${}^{n-1}Q_r={}^{n-2}Q_r+\ .\ .\ .\ \text{ writing } (n-1) \text{ for } n,$$

$${}^{n-2}Q_r={}^{n-3}Q_r+\ .\ .\ .$$

$$.\quad .\quad .\quad .$$

$${}^{2}Q_r={}^{1}Q_r+\ .\ .\ .$$

$\therefore$ by addition ${}^{n}Q_r={}^{1}Q_r+$ an integer, but ${}^{1}Q_r=1$;

$\therefore$ ${}^{n}Q_r$ is integral.

$\therefore$ If the law hold for $(r-1)$ consecutive integers, it holds also for r. But it does hold for 2; therefore for 3, and so by mathematical induction universally.

Example.—Prove $4n^3-n$ divisible by 3.

$$4n^3-n=n(4n^2-1)=n(2n-1)(2n+1)=\frac{(2n-1)2n(2n+1)}{2}.$$

Now $(2n-1)2n(3n+1)$ is divisible by $|\underline{3}$;

$\therefore\ (2n-1)2n(2n+1)=6p$ say, where p is integral;

$$\therefore\ \frac{(2n-1)2n(2n+1)}{2}=3p,$$

whence $4n^3-n$ is divisible by 3.

Aliter.—n must be of the form $3m$ or $3m\pm1$.

Hence (i) if $n=3m$ the proposition is obvious,

(ii) if $n=3m\pm1$,

$$4n^3-n=4(3m\pm1)^3-n=4(27m^3\pm27m^2+9m\pm1)-(3m+1),$$

and this of the form $3k\pm4\mp1$,

$=3k\pm3$,

and therefore is divisible by 3.

145. Every square number is of the form $5m$ or $5m\pm1$.

For *every* number being of the form $5m$, $5m\pm1$, $5m\pm2$, every square number is of the form

$$25m^2,\quad 25m^2\pm10m+1,\quad 25m^2\pm20m+4.$$

Now these may be written respectively,

$$5(5m^2),\quad 5(5m^2\pm2m)+1,\quad 5(5m^2\pm2m+1)-1.$$

Hence every square number is of the form

$$5m \text{ or } 5m\pm1.$$

Cor.—No square number ends with 2, 3, 7, or 8.

For from the above every square number ends with

0 5 1 6 9 or 4.

Example. If $a^2+b^2=c^2$, prove that either a, b, or c is divisible by 5.

Suppose neither a nor b divisible by 5.

Hence they are of the form $5m\pm1$ or $5m\pm2$, and therefore their squares of the form $5m\pm1$;

$$\therefore\ c^2=a^2+b^2=5p\pm1+5q\pm1.$$

Now the terms with double signs must cancel, or we should have a square of the form $5M \pm 2$, which is impossible;

$$\therefore\ c^2 \text{ and consequently } c \text{ is of the form } 5M,$$

whence c is divisible by 5. Similarly for any case.

146. Every prime number greater than 3 is of the form $6m \pm 1$.

For every number is of the form

$$6m,\quad 6m \pm 1,\quad 6m \pm 2, \text{ or } 6m+3,$$

And the primes cannot be of the form

$$6m,\quad 6m \pm 2, \text{ or } 6m+3.$$

Therefore they *are* of the form $6m \pm 1$.

147. *Every number which is not a prime is divisible by some number not greater than its square root.*

For let p be any number which is not a prime.

Therefore it must be separable into two factors at least, a and b say;

$$\therefore\ p = ab;$$

$$\therefore\ \sqrt{p}\,\sqrt{p} = ab.$$

$$\begin{aligned} \text{Hence if } a &> \sqrt{p}, & b &> \sqrt{p}, \\ \text{if } a &< \sqrt{p}, & b &> \sqrt{p}, \\ \text{if } a &= \sqrt{p}, & b &= \sqrt{p}. \end{aligned}$$

Therefore in any case there is a factor not greater than the square root.

148. *No Algebraical formula can express primes only.*

If possible, let the formula $a+bx+cx^2+\ .\ .\ .$, denote primes for all values of x, and let P, Q, be the resulting primes when $x=m$, and $m+nP$ respectively;

$$\begin{aligned} \therefore\ Q &= a+b(m+nP)+c(m+nP)^2+\ .\ .\ . \\ &= a+bm+cm^2+\ .\ .\ .\ +kP \text{ say,} \\ &= P+kP, \\ &= P(1+k), \end{aligned}$$

i.e. the *prime* Q is separable into factors other than itself and unity, which is impossible.

Therefore no Algebraical formula can express primes only.

Bishop Colenso notices that the expression $2x^2+29$ gives 29 primes as x receives the value 0, 1, 2, . . . but that beyond that point, in accordance with the theorem above, it ceases to give prime numbers only.

149. *The number of primes is indefinitely great.*

For if not let p be the greatest, so that $2.3.5.7 \ldots p$ is the product of *all existing primes.* Denote this product by P; $\therefore$ P is divisible by every one of these primes, and $P+1$ by none of them. Hence $P+1$ is either a prime, or divisible by some prime greater than p. In either case the supposition that p is the greatest prime is untenable. $\therefore$ etc.

150. *If* c *measures* ab, *but is prime to* a, *it measures* b.

For take $a>c$ and perform the operation of finding the G.C.M. of a and c.

Let the quotients so obtained be $q_1, q_2, q_3, \ldots$ and the remainders $r_1, r_2, r_3, \ldots 1$. (For since a *is prime to* c we must by proceeding far enough obtain a remainder 1.) Thus

$$a=q_1c+r_1, \quad \ldots \quad \text{(i)},$$
$$c=q_2r_1+r_2, \quad \ldots \quad \text{(ii)},$$
$$r_1=q_3r_2+r_3, \quad \ldots \quad \text{(iii)},$$
$$\ldots\ldots$$

Multiply each of these by b and divide by c,

$$\therefore \frac{ab}{c}=bq_1+\frac{br_1}{c}, \quad \ldots \quad \text{(i)},$$
$$b=\frac{br_1}{c}.q_2+\frac{br_2}{c}, \quad \ldots \quad \text{(ii)},$$
$$\frac{br_1}{c}=\frac{br_2}{c}.q_3+\frac{br_3}{c}, \quad \ldots \quad \text{(iii)},$$
$$\ldots\ldots$$

Then since $\frac{ab}{c}$ is integral by hypothesis, it follows from (i) that $\frac{br_1}{c}$ is likewise integral. Hence from (ii) $\frac{br_2}{c}$ is integral, and so finally $\frac{b \times 1}{c}$ is integral.

Similarly the proof holds if $c > a$.

151. *If a and b are prime to c, ab is prime to c.*

For if not, if possible let ab and c have a common factor k.

Now a is prime to c and therefore to k.

And since k is prime to a, and measures ab, . . (Hyp.)

$\therefore$ k measures b, ... (150),

$\therefore$ b and c have a common factor k,

and they are prime to each other, . (Hyp.)

Which is absurd. Therefore, etc.

152. *If a be prime to b, the fraction $\frac{a}{b}$ cannot be expressed in lower terms.*

For if possible let $\frac{a}{b} = \frac{c}{d}$ where c and d are less than a and b respectively.

Then since $\frac{ad}{b} = c =$ an integer,

and b is prime to a, . . (Hyp.)

$\therefore$ b measures d.

But b is $> d$, ... (Hyp.)

Which is absurd. Therefore, etc.

Cor.—If $\frac{a}{b} = \frac{c}{d}$, a being prime to b, c must be the same multiple of a, that d is of b.

For since $\frac{a}{b} = \frac{c}{d}$; $\therefore$ $\frac{ad}{b} = c$,

and a is prime to b; $\therefore$ b measures d; $\therefore$ $d = mb$ say,

whence $c = \frac{a.mb}{b} = ma$. Q.E.D.

153. *If m be prime to n, the remainders when r, $m+r$, $2m+r$, $3m+r$, . . . $(\overline{n-1}|m+r)$ are divided by n, will all be different.*

For if not, if possible let two of them, $pm+r$ and $qm+r$ give the same remainder, and let

$$pm+r=hn+t,$$
$$\text{and } qm+r=kn+t;$$
$$\therefore (p-q)m=(h-k)n;$$
$$\therefore \frac{m}{n}=\frac{h-k}{p-q}.$$

Now m is prime to n, ... (Hyp.)

$\therefore (p-q)$ is a multiple of n.

But p and q are each less than n,

Which is absurd. Therefore the remainders *are* all different.

Cor. (i)—The n remainders are obviously though not necessarily in order 0, 1, 2, 3 . . . $(n-1)$.

Cor. (ii)—When the remainder is prime to n, so also is the dividend. Hence if we denote the number of integers less than n and prime to n by $P(n)$, it follows that $P(n)$ of these remainders, and therefore $P(n)$ of the given quantities are prime to n.

154. *If m be prime to n, $P(mn)=P(m)\times P(n)$.*

For if 1, a, b . . . r . . . $m-1$ are the numbers less than m and prime to m, we have for all the numbers less than mn and prime to m.

$$\left|\begin{array}{r|r|r|c|r|c|r} 1 & a & b & \ldots & r & \ldots & m-1 \\ m+1 & m+a & m+b & \ldots & m+r & \ldots & 2m-1 \\ 2m+1 & 2m+a & 2m+b & \ldots & 2m+r & \ldots & 3m-1 \\ & & & & & & \\ (n-1)m+1 & (n-1)m+a & (n-1)m+b & \ldots & (n-1)m+r & \ldots & nm-1 \end{array}\right|.$$

There being $P(m)$ quantities in each *row* and n quantities in each *column.*

But those numbers which are prime to mn are prime to n as well as to m (151).

Hence to find $P(mn)$ we must select from the preceding table those numbers which are prime to n.

Now in each column there are $P(n)$ such numbers (152, Cor. ii), and there are $P(m)$ columns;

$$\therefore\ P(mn)=P(m)\times P(n).$$

155. **To find** $P(N)$.

Let $N=a^pb^qc^r\ldots$ where $a, b, c, \ldots$ are prime numbers.

Then $P(N)=P(a^p).P(b^q)P(c^r)\ldots \quad . \quad . \quad . \quad (153).$

Now consider the series $1, 2, 3 \ldots a^p$.

The only terms here which are *not* prime to a^p are

$$a,\ \ 2a,\ \ 3a,\ \ a^{p-1}.a,$$

the number of which is a^{p-1};

$\therefore$ the number which *are* prime to $a^p=a^p-a^{p-1}$.

$$\text{Hence } P(a^p)=a^p\left(1-\frac{1}{a}\right),$$

$$\text{so } P(b^q)=b^q\left(1-\frac{1}{b}\right),$$

$$\text{and } P(c^r)=c^r\left(1-\frac{1}{c}\right),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\therefore\ P(N)=a^pb^qc^r\ldots\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots$$

$$=N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots$$

156. *To find the number of divisors of a given number.*

Let N be the given number, and let it $=a^pb^qc^r\ldots$ $a, b, c\ldots$ being prime numbers.

Then obviously the terms of the product

$$(1+a+a^2+\ldots+a^p)(1+b+b^2+\ldots+b^q)(1+c+c^2+\ldots+c^r)\ldots$$

involve all the possible divisors *without excess or defect.*

Now the number of terms in these brackets is

$$(p+1),\ (q+1),\ (r+1)\ .\ .\ .\ \text{respectively};$$

$\therefore$ the number of divisors required is $(p+1)(q+1)(r+1)\ .\ .\ .$

Cor.—It is plain that the *sum* of the divisors is the *product itself*, and this may be written

$$\frac{a^{p+1}-1}{a-1}\cdot\frac{a^{q+1}-1}{b-1}\cdot\frac{c^{r+1}-1}{c-1}\ .\ .\ .$$

157. *If m be a prime number every term in the expansion of $(a+x)^m$, except the first and last, will have a coefficient of the form mP where P is integral.*

For each of these coefficients is of the form

$$\frac{m(m-1)\ .\ .\ .\ (m-r+1)}{\underline{|r}}.$$

Now (i) this is *integral*, for it is the product of r consecutive integers divided by $\underline{|r}$.

And (ii) m is a factor of that integer; for being a prime, and being greater than r, it cannot cancel out with any factor in the denominator.

Hence the coefficient is of the form mP where P is integral.

Cor.—Similarly, it is clear that every term in the expansion of $(a+b+c+\ .\ .\ .)^m$ except a^m, b^m, c^m, . . . have coefficients of the form mP where P is integral.

158. **FERMAT'S THEOREM. If m be a prime number, and prime to N, then $N^{m-1}-1$ is divisible by m.**

For let a, b, c, . . . denote any integers. Then $(a+b+c+\ .\ .\ .)^m=a^m+b^m+c^m+\ .\ .\ .\ +mh$, where h is integral.

For (157, Cor.) the coefficients are integral and involve m, and a, b, c, . . . and their powers are integral by assumption.

Now let there be N of these quantities $a, b, c, \ldots$ Take each of them equal to 1, and let h become k when this substitution is made;

$$\therefore\ N^m = N + mk, \text{ where } k \text{ is integral};$$
$$\therefore\ N(N^{m-1} - 1) = mk;$$
$$\therefore\ m \text{ measures } N(N^{m-1} - 1),$$
$$\text{and it is prime to } N; \quad . \quad . \quad \text{(Hyp.)}$$
$$m \text{ measures } N^{m-1} - 1, \quad . \quad . \quad (150).$$

This is generally expressed thus:

$$N^{m-1} = 1 + pm, \text{ where } p \text{ is integral.}$$

159. **LOGARITHMS. The logarithm of a number to a given base is the index of the power to which the base must be raised in order to be equal to the given number.**

Thus $10^2 = 100$ and 2 is the log of 100 to base 10.

This is written thus, $\log_{10} 100 = 2$.

160. *The log of the base itself is* 1.

For $a^1 = a$ always.

The log of 1 *is* 0, *whatever be the base.*

For $a^0 = 1$ always.

161. *The log of a product is the sum of the logs of the factors.*

For let $\log_a m = x$, and $\log_a n = y$;

$$\therefore\ m = a^x, \qquad \text{and } n = a^y;$$
$$\therefore\ mn = a^{x+y};$$
$$\therefore\ \log_a mn = x + y = \log_a m + \log_a n.$$

162. *The log of a quotient is the log of the dividend diminished by the log of the divisor.*

For with the same assumptions as in the last article

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y};$$

$$\therefore\ \log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

163. *To connect $\log_a m$ and $\log_b m$.*

Let $\log_a m = x$, and $\log_b m = y$;

$$\therefore\ m = a^x = b^y;$$

$$\therefore\ b^{\frac{y}{x}} = a;$$

$$\therefore\ \frac{y}{x} = \log_b a;$$

$$\therefore\ x = y \times \frac{1}{\log_b a};$$

$$\therefore\ \log_a m = \log_b m \times \frac{1}{\log_b a}.$$

Cor.—Since $\log_b a = \dfrac{y}{x}$, and similarly $\log_a b = \dfrac{x}{y}$;

$$\therefore\ \log_b a \times \log_a b = 1.$$

164. In all theoretical investigations the base adopted is the series $\left(1+1+\dfrac{1}{\underline{|2}}+\dfrac{1}{\underline{|3}}+\ \ldots\ \textit{ad inf.}\right)$. This sum is denoted by the symbol e. It lies between 2 and 3, being evidently greater than 2 and less than $1+1+\dfrac{1}{2}+\dfrac{1}{2^2}+\ \ldots$ *i.e.* <3. To this base logarithms were *originally* calculated. It may be easily proved *incommensurable.*

165. **EXPONENTIAL THEOREM. To prove that**

$$a^x = 1 + (\log_e a)x + \frac{(\log_e a)^2 x^2}{\underline{|2}} + \frac{(\log_e a)^3 x^3}{\underline{|3}} + \ \ldots$$

$$\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^n\right\}^x \text{ for all values of } n;$$

$$\therefore\ 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}} + \ \ldots$$

$$= \left\{1 + 1 + \frac{1-\frac{1}{n}}{\underline{|2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}} + \ \ldots\right\}^x.$$

Now let n be indefinitely increased and we have

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots=\left(1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\right)^x=e^x.$$

For in each term we have left out a part which ultimately vanishes; but it is not evident that an infinite number of such small parts vanish; as, however, in each term the *part rejected* is infinitely small in comparison with the *part retained*, and all the parts retained *have the same sign*, therefore the sum of the parts rejected is infinitely small in comparison with the sum of the parts retained.

Now let $\log_e a=k$, so that $a=e^k$.

Then since $e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots$

$$\therefore\ e^{kx}=1+kx+\frac{k^2x^2}{\underline{|2}}+\frac{k^3x^3}{\underline{|3}}+\ldots$$

$$\therefore\ a^x=1+(\log_e a)x+\frac{(\log_e a)^2x^2}{\underline{|2}}+\ldots$$

This result is called the Exponential Theorem.

166. **To prove** $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-$

Since $a^x=1+(\log_e a)x+\frac{(\log_e a)^2x^2}{\underline{|2}}+\ldots$

$$\therefore\ a^y=1+(\log_e a)y+\frac{(\log_e a)^2y^2}{\underline{|2}}+\ldots$$

In the last result write $(1+x)$ for a, transpose and divide both sides by y;

$$\therefore\ \frac{(1+x)^y-1}{y}=\log_e(1+x)+\text{ terms involving } y,\ y^2\ldots$$

$$\therefore\ x+\frac{y-1}{\underline{|2}}.x^2+\frac{(y-1)(y-2)}{\underline{|3}}x^3+\ldots=\log_e(1+x)+\ldots$$

Equate terms independent of y, and we have

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$$

Note.—In the preceding proof the series $1+(\log_e a)x+\ldots$ is convergent, and x *must be taken* <1, so that

$$x+\frac{y-1}{\lfloor 2}x^2+\frac{(y-1)(y-2)}{\lfloor 3}x^3+\ldots \text{ is so likewise.}$$

167. Since $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$ *when* $x<1$;

$$\therefore \log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots \quad ,,$$

$$\therefore \log\frac{1+x}{1-x}=2(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots) \quad ,,$$

$$\text{Let } \frac{1+x}{1-x}=\frac{n+1}{n}; \qquad \therefore x=\frac{1}{2n+1},$$

and we have $\log\frac{n+1}{n}=2\left[\frac{1}{2n+1}+\frac{1}{3}\left(\frac{1}{2n+1}\right)^3+\ldots\right]$

for all values of n. Put $n=1$;

$$\therefore \log 2-\log 1=2\left[\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}\right)+\ldots\right].$$

But $\log 1=0$;

$$\therefore \log 2=2\left[\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}\right)^3+\ldots\right].$$

So now if we put $n=2$ we shall obtain $\log 3$ = a series $+\log 2$; and so generally we shall obtain the log of *any number*, if the log of the preceding number be known.

If in the above we had written $\frac{n^2}{n^2-1}$ for $\frac{1+x}{1-x}$ we should have obtained a still more rapidly converging series for ascertaining the log of any number. In this case we are required, however, to know the logs of the *two* preceding numbers.

The student should verify this for himself.

For further information on logarithms he is referred to the chapters on that subject in Mr. Todhunter's *Plane Trigonometry*.

168. *If n be a positive integer then*

$$n^n-\frac{n}{1}(n-1)^n+\frac{n(n-1)}{1.2}(n-2)^n- \ldots =\underline{|n}.$$

For by the Binomial theorem

$$(e^x-1)^n=e^{nx}-\frac{n}{1}e^{(n-1)x}+\frac{n(n-1)}{1.2}e^{(n-2)x}- \ldots$$

And by the Exponential theorem

$$(e^x-1)^n=(1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+ \ldots-1)^n=(x+\frac{x^2}{\underline{|2}}+ \ldots)^n.$$

$=x^n+$ terms involving higher powers of x than the nth.

Hence equating coefficients of x^n, in these two equivalent expressions we have

$$\frac{n^n}{\underline{|n}}-\frac{n}{1}\frac{(n-1)^n}{\underline{|n}}+\frac{n.n-1}{1.2}\frac{(n-2)^n}{\underline{|n}}- \ldots =1;$$

$$\therefore n^n-\frac{n}{1}(n-1)^n+\frac{n.(n-1)}{1.2}(n-2)^n- \ldots =\underline{|n}.$$

169. **WILSON'S THEOREM. If n be a prime number $1+\underline{|n-1}$ is divisible by n.**

By the preceding article

$$(n-1)^{n-1}-\frac{n-1}{1}(n-2)^{n-1}+\frac{(n-1)(n-2)}{1.2}(n-3)^{n-1}-\ldots=\underline{|n-1}.$$

Now n is a prime, and prime to $n-1$;

$\therefore\ (n-1)^{n-1}=1+pn$ by Fermat's theorem,

so $(n-2)^{n-1}=1+qn$ „

and $(n-3)^{n-1}=1+rn$ „

. . . .

$$\therefore\ (1+pn)-\frac{n-1}{1}(1+qn)+\frac{(n-1)(n-2)}{1.2}(1+rn)-\ldots=\underline{|n-1};$$

$$\therefore\ \underline{|n-1}=kn+[1-\frac{n-1}{1}+\frac{(n-1)(n-2)}{1.2}-\ldots\text{to }(n-1)\text{ terms}],$$

where k is some integer, for $p, q, r,$...are integers,

$=kn+[(1-1)^{n-1}-(-1)^{n-1}]$,

$=kn-1$ for *n being a prime $n-1$ is even;*

$\therefore\ 1+\underline{|n-1}=kn$, and consequently is divisible by n.

Note.—If n be not a prime $1+\underline{|n-1}$ is not divisible by n: for let a be a factor of n; $\therefore$ $\underline{|n-1}$ is divisible by a; $\therefore$ $1+\underline{|n-1}$ is *not* divisible by a, and $\therefore$ not by n.

CONVERGENCE AND DIVERGENCE.

170. A series continued *ad inf.* is said to be convergent when the sum of the first n terms cannot exceed some fixed finite quantity, and divergent when the sum of the first n terms may be made greater than *any* finite quantity, by making n sufficiently large.

171. **Tests of Convergence.**

(All the terms are considered positive finite quantities unless the contrary is specified.)

(i) *The series is convergent if from and after some particular term the ratio of each term to the preceding one is less than some fixed quantity* (k) *which is itself* <1.

Let the particular term, and those immediately following it, be $u_1, u_2, u_3, \ldots$ Let s denote the sum of the preceding terms, and S that of the whole series. Thus

$$S=s+u_1+u_2+u_3+\ \ldots\ ad\ inf.,$$

$$=s+u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\frac{u_2}{u_1}+\frac{u_4}{u_3}\frac{u_3}{u_2}\frac{u_2}{u_1}+\ \ldots\ ad\ inf.\right),$$

$$<s+u_1(1+k+k^2+\ \ldots\ ad\ inf.),$$

$$<s+u_1\frac{1}{1-k}\text{ since }k<1.$$

Now each term is finite; $\therefore$ s is necessarily finite;

$\therefore$ S is less than some fixed finite quantity;

$\therefore$ the series is convergent.

(ii) *A series is convergent when from and after some particular term each term is less than the corresponding term of another series which is known to be convergent.*

This is evident from the meaning of "convergence."

(iii) *A series is convergent if the terms are alternately positive and negative, and continually decrease numerically.*

Let the series be $u_1-u_2+u_3-u_4+\ .\ .\ .$ and let the sum be denoted by S.

Then since $S=(u_1-u_2)+(u_3-u_4)+\ .\ .\ .$

or $\quad u_1-(u_2-u_3)-(u_4-u_3)-\ .\ .\ .$

and all the quantities (u_1-u_2), (u_3-u_4), $(u_2-u_3)\ .\ .\ .$ are evidently positive, we have

$$S>u_1 \text{ and} <u_1-u_2\,;$$

$\therefore$ the series is convergent.

Tests of Divergence.

(iv) *A series is divergent when from and after some fixed term each term is greater than the preceding one.*

Let the particular term and those immediately following it be $u_1, u_1, \ldots$: s the sum of the preceding terms, and S that of the whole series.

Then $S=s+u_1+u_2+u_3+\ .\ .\ .$

$<s+u_1+u_1+u_1+\ .\ .\ .$

$<s+mu_1$ say.

Now by sufficiently increasing m, mu_1 may be made greater than any finite quantity whatever;

$\therefore$ the series is divergent.

Cor.—It is clear that the series is likewise divergent if $u_1=u_2=u_3=\ .\ .\ .$

Hence we may say that *a series is divergent if from and after any particular term the ratio of any term to the preceding one is equal to or greater than* 1.

(v) *A series is divergent if from and after some fixed term each term is greater than the corresponding term of a series which is known to be divergent.*

This is evident from the meaning of "divergence."

172. Examples of the application of these tests.

(i) $1+\dfrac{1}{\underline{|2}}+\dfrac{1}{\underline{|3}}+\ .\ .\ .$ is convergent.

For $\dfrac{\text{the } (n+1)\text{th term}}{\text{the } n\text{th term}}=\dfrac{\dfrac{1}{\lfloor n+1}}{\dfrac{1}{\lfloor n}}=\dfrac{1}{n+1}$.

Hence if $n=1$ the ratio is $\dfrac{1}{2}$, but from and beyond the second term the ratio is $<\dfrac{1}{2}$;

$\therefore$ the series is convergent.

(ii) $\dfrac{1}{2.3}+\dfrac{1}{4.5}+\dfrac{1}{6.7}+\ldots$ is convergent.

For this series $=\dfrac{1}{2}-\dfrac{1}{3}+\dfrac{1}{4}-\dfrac{1}{5}+\ldots$ and is thus convergent by the third test.

(iii) $1+\dfrac{1}{\lfloor 2}+\dfrac{1}{\lfloor 3}+\ldots$ is convergent.

For $1+\dfrac{1}{1.2}+\dfrac{1}{1.2.3}+\dfrac{1}{1.2.3.4}+\ldots$ is

$<1+\dfrac{1}{2}+\dfrac{1}{2.2}+\dfrac{1}{2.2.2}+\ldots$ evidently

$<1+\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\ldots$

$<\dfrac{1}{1-\frac{1}{2}}$ and is therefore convergent.

(iv) $1+\dfrac{1}{2^2}+\dfrac{2^3}{3^3}+\dfrac{3^3}{4^4}$ is divergent.

For $\dfrac{\text{the } (n+1)\text{th term}}{\text{the } n\text{th term}}=\dfrac{\dfrac{n^2}{(n+1)^{n+1}}}{\dfrac{(n-1)^{n-1}}{n^n}}=\dfrac{n^{2n}}{(n+1)^{n+1}(n-1)^{n-1}}$,

$$=\frac{n^{n+1}.n^{n-1}}{(n+1)^{n+1}(n-1)^{n-1}}=\left(\frac{n}{n+1}\right)^{n+1}\times\left(\frac{n-1}{n}\right)^{1-n},$$

$$=\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}\times\left(1-\frac{1}{n}\right)^{1-n},$$

$$=\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{-n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{n}}.$$

Now when n is indefinitely increased the limits of $\left(1-\frac{1}{n}\right)$ and $\left(1+\frac{1}{n}\right)$ are both unity: and those of $\left(1-\frac{1}{n}\right)^{-n}$ and $\left(1+\frac{1}{n}\right)^{n}$ are both e;

$\therefore$ when n is indefinitely increased the ratio of the $(n+1)$th term to the nth term is 1;

$\therefore$ the series is divergent.

(v) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \ldots$ is divergent.

For this may be written

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)+\cdots$$

the first bracket containing two terms; the second four; the third eight; and so on.

Hence writing for each fraction in each bracket, the smallest, *i.e.* the last, fraction included in that bracket, we have the given series:

$$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\ \ldots$$

$$>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ \ldots$$

$$>1+\left(m\times\frac{1}{2}\right)\text{ say.}$$

Now by taking m large enough this may be made greater than *any* finite quantity;

$\therefore$ the series is divergent.

173. It may happen that the ratio of the $(n+1)$th term to the nth term, though always less than 1, tends to 1 as its limit. In this case we cannot name any proper fraction k which is always in excess of this ratio, and therefore the first test of convergency is here useless. The example last given is a case in point, and the method of treatment adopted there is typical. As a further illustration we will examine the convergence or divergence of the series.

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\ \cdot\ \cdot\ \cdot$$

(i) When $p=1$ the series has been already proved divergent,

(ii) When $p>1$ the series is convergent.

For $S=1+\frac{1}{2^p}+\frac{1}{3^p}+\ \cdot\ \cdot\ \cdot$

$$<1+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)+\ \cdot\ \cdot\ \cdot$$

$$<1+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\ \cdot\ \cdot\ \cdot\ \textit{ad inf.}$$

$$<\frac{1}{1-\frac{1}{2^{p-1}}}\ \text{for}\ p>1.$$

and therefore the series is convergent.

(iii) When $p<1$ the series is divergent: for each term of the series except the first is greater than the corresponding term of the series $1+\frac{1}{2}+\frac{1}{3}+\ \cdot\ \cdot\ \cdot$ which is known to be divergent.

174. **Further Tests.**

(i) *A series is convergent when from and after some particular term the ratio of each term to the preceding is less than the corresponding ratio in another series which is known to be convergent.*

This follows from the first test of 171.

(ii) *A series is divergent when from and after some particular*

term the ratio of each term to the preceding is greater than the corresponding ratio in another series which is known to be divergent.

This follows from the fourth test of 171.

(iii) If the ratio of the $(n+1)$th term to the nth $=\dfrac{1}{1+k}$, and when n is indefinitely increased, nk remains finite, but $\dfrac{1}{1+k}$ has 1 for its limit, then if from and after some particular term $nk > \dfrac{p}{q}$ (an improper fraction) the series is convergent, but if $nk<1$ the series is divergent.

(a) Let $nk > \dfrac{p}{q}$ after some particular term.

Then $(1+k)^q > \left(1+\dfrac{1}{n}\right)^p$,

if $qk+\dfrac{q(q-1)}{12}k^2+\ldots > p.\dfrac{1}{n}+\dfrac{p.p-1}{1.2}\dfrac{1}{n^2}+\ldots$

if $nk+\left(nk\dfrac{q-1}{2}k+\ldots\right) > \dfrac{p}{q}+\left(\dfrac{p}{q}.\dfrac{p-1}{1.2}\dfrac{1}{n}+\ldots\right)$,

if $nk-\dfrac{p}{q} > A-B$ say.

Now nk is finite always, and the limit of k is 0; $\therefore$ by indefinitely increasing n, A and B and therefore their difference can be made as small as we please; and $\therefore$ less than $nk-\dfrac{p}{q}$.

Hence $(1+k)^q$ *is* $> \left(1+\dfrac{1}{n}\right)^p$;

$$\therefore 1+k > \left(1+\frac{1}{n}\right)^{\frac{p}{q}};$$

$$\therefore \frac{1}{1+k} < \frac{1}{\left(1+\frac{1}{n}\right)^{\frac{p}{q}}} < \frac{\dfrac{1}{(n+1)^{\frac{p}{q}}}}{\dfrac{1}{n^{\frac{p}{q}}}};$$

that is, the ratio of the $(n+1)$th term of the given series to its nth term is $<$ the ratio of the $(n+1)$th term of the series $1+\frac{1}{2^m}+\frac{1}{3^m}+\ldots$ to *its* nth term, m being >1: but the latter series is convergent; $\therefore$ so also is the given series.

(β) Let $nk<1$;

$$\therefore k<\frac{1}{n};$$

$$\therefore 1+k<1+\frac{1}{n}<\frac{n+1}{n};$$

$$\therefore \frac{1}{1+k}>\frac{n}{n+1}>\frac{\frac{1}{n+1}}{\frac{1}{n}}.$$

That is, the ratio of the $(n+1)$th term of the given series to its nth term is $>$the ratio of the $(n+1)$th term of the series $1+\frac{1}{2}+\frac{1}{3}+\ldots$ to *its* nth term.

But the latter series is known to be divergent; $\therefore$ so also is the given series.

Example. The series $1+1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1.3}{2.4}\cdot\frac{1}{5}+\ldots$ is convergent.

For it may be easily shown that the ratio of the $(n+1)$th term to the nth may be written

$$\frac{1}{1+\frac{6n-7}{4n^2-12n+9}};$$

$$\therefore nk=\frac{6n^2-7n}{4n^2-12n+9}=\frac{6-\frac{7}{n}}{4-\frac{12}{n}+\frac{9}{n^2}}.$$

Hence the limiting value of $nk=\frac{3}{2}$ when n is indefinitely increased; and is $>\frac{6}{4}$ if

$$24n^2-28n>24n^2-72n+54,$$

if $$44n>54,$$

f $$n>\tfrac{54}{44};$$

$\therefore$ from and after the second term $nk>\frac{3}{2}$

$\therefore$ the series is convergent.

RECURRING SERIES.

175. If from and after some particular term, each term of a decreasing *infinite* series be formed by some fixed law from the preceding terms, the series is called a recurring series, and is said to be of the first, second, third, . . . order, according as the law involves one, two, three, . . . of the preceding terms.

To find the sum of a recurring series of the second order.

Let the series be $a+bx+cx^2+dx^3+\ldots$; denote the several terms by A, B, C, D, . . . the sum by S, and let $c=fxB+gx^2A$, where $f+g$ is called the scale of relation.

Now
$$A=A,$$
$$B=B,$$
$$C=fxB+gx^2A,$$
$$D=fxC+gx^2B,$$
$$\cdots\cdots$$
$$\therefore\ S=A+B+fx(S-A)+gx^2S;$$
$$\therefore\ S=\frac{A+B-fxA}{1-fx-gx^2}.$$

Similarly we may show that the sum of a recurring series of the third order (the scale of relation being $f+g+h$) is

$$\frac{A+B+C-fx(A+B)-gx^2A}{1-fx-gx^2-hx^3},$$

and the student will easily see from this how to express the sum of a recurring series of the nth order.

Examples.

(i) Find the sum of $1+x+3x^2+7x^3+17x^4+\ldots$

Here the scale of relation is evidently $2+1$;

$$\therefore\ S=\frac{1+x-2x}{1-2x-x^2}=\frac{1-x}{1-2x-x^2}.$$

(ii) Find the sum of $1+x+x^2+3x^3+5x^4+9x^5+\ldots$

Assume the scale of relation $f+g+h$;

$$\therefore\ f+g+h=3,\quad 3f+g+h=5,\quad 5f+3g+h=9,$$

whence $f=1$, $g=1$, and $h=1$;

$$\therefore\ S=\frac{1+x+x^2-x(1+x)-x^2(1)}{1-x-x^2-x^3},$$

$$=\frac{1-x^2}{1-x-x^2-x^3}.$$

INDETERMINATE EQUATIONS.

176. If we have one equation given between two unknowns, each of those unknowns may have an infinite number of values, for we may give any value we please to one of them, and then find the corresponding value of the other so as to satisfy the equation. If, however, we restrict the roots to be positive integers (and not zero), *as we shall throughout this chapter*, the number of solutions will *often* be limited.

FORM. $ax+by=c$, a, b, **and** c **being positive integers.**

(i) *a must be prime to b.*

For if not, let them have a common factor r.

Then this cannot appear in c (for we assume, of course, that the equation is in simplest form).

Let $a=pr$ and $b=qr$, p, q, r, being necessarily positive integers. Then

$$prx+qry=c;\qquad \therefore\ px+qy=\frac{c}{r}=\text{a fraction},$$

and p and q are positive integers;

$\therefore$ x or y is fractional.

And this is contrary to the restriction; $\therefore$ *a is prime to b.*

(ii) *If h, k be one pair of roots all the others can be found.*

For $ax+by=c=ah+bk$;

$\therefore\ a(x-h)=b(k-y)$;

$\therefore\ \dfrac{a}{b}=\dfrac{k-y}{x-h}$, and a is prime to b;

$\therefore\ k-y$ is the same multiple of a that $x-h$ is of b.

$\therefore\ at=k-y$ and $bt=x-h$, *where t is integral.*

$\therefore\ x=h+bt$ and $y=k-at$.

Now since y is a positive integer t cannot be taken positively greater than $\dfrac{k}{a}$, and since x is a positive integer t cannot be taken negatively less than $-\dfrac{h}{b}$.

The values which t is capable of receiving are thus limited in *both directions*, and the number of positive integral solutions of the equation is thus limited also.

Cor.—It is evident that the roots are in A.P.

(iii) *The roots may be found by the method of continued fractions.*

For let $\dfrac{n}{d}$ be the convergent immediately preceding $\dfrac{a}{b}$;

$$\therefore\ ad-bn=\pm 1.$$

Take $ad-bn=+1$; $\qquad \therefore\ adc-bnc=c,$

$$\text{and } ax+by=c;$$

$$\therefore\ a(x-dc)+b(y+nc)=0.$$

This is clearly satisfied by $x-dc=bt$, and $y+nc=-at$;

$$\therefore\ x=dc+bt, \text{ and } y=nc+at.$$

Similarly we may deal with the other case.

(iv) *The number of solutions cannot exceed* $\dfrac{c}{ab}+1$.

For t cannot be greater than $\dfrac{k}{a}$ nor algebraically less than

$-\frac{h}{b}$. Hence counting in the value $t=0$ there cannot be more than

$$\frac{k}{a}+\frac{h}{b}+1 \text{ solutions.}$$

But this $=\frac{ah+bk}{ab}+1$, and $ah+bk=c$;

$\therefore$ there cannot be more than $\frac{c}{ab}+1$ solutions.

(v) $a+b$ cannot be $>c$.

Let x and y each have their *least* admissible value, viz. 1.

Then $ax+by=a+b$,

$>c$, and *à fortiori* in all other cases;

$\therefore$ the equation $ax+by=c$ has no positive integral solutions.

FORM. $ax-by=c$.

177. Modification of the preceding results:—

(i) Holds equally for this form.

(ii) We obtain $x=h+bt$, $y=k+at$.

Hence t is unrestricted in magnitude *in the positive direction*, and the number of roots will be unlimited.

(iii) The same *method* of reasoning holds.

178. If we have one equation between three unknowns, say $ax+by+cz=d$, we must write

$$ax+by=d-cz.$$

Ascribe to z the values 1, 2, 3, . . . and obtain thus a series of equations of the form $ax+by=c$.

179. We add two examples of the method of working practically.

(i) Solve $7x+15y=225$ in positive integers,

$$x+y+\frac{y}{7}=32+\frac{1}{7};$$

$$\therefore \frac{y-1}{2}=32-x-y-=\text{an integer}=t \text{ say};$$

$$\therefore y=1+7t, \text{ whence } x=30-15t.$$

Now as we exclude zero and negative values for x and y, t is $>-\frac{1}{7}$ and <2. The only values which it can receive are 0.1, whence $x=30$ or 15, and $y=1$ or 8.

(ii) Solve $xy+x^2=x+3y+23$ in positive integers;

$$\therefore\ y(x-3)=x+14-(x^2-9),$$

$$y=\frac{x+14}{x-3}-x-3=1+\frac{17}{x-3}-x-3,$$

$$=\frac{17}{x-3}-(x+2).$$

Now y being a positive integer $\frac{17}{x-3}$ must be integral, and $\therefore\ x-3=1$ or 17. The latter value is inadmissible, as we should have $y=1-x-2$, which could not be positive for positive values of x;

$$\therefore\ x-3=1;\qquad \therefore\ x=4,$$

$$\text{and } y=17-6=11.$$

INTEREST AND DISCOUNT.

180. **Simple Interest.** *If M be the amount of P£ for n years; and r the interest on one pound for one year, $M=P(1+nr)$.*

For the interest on £1 for a year being r,

„	P	„	is	Pr,
„		n years is		Pnr;

$$\therefore\ M=P+Pnr=P(1+nr).$$

181. **Compound Interest.** *If M be the amount of P£ for n years; and R the amount of £1 for a year, $M=PR^n$.*

The amount of £1 for a year is R, and the amount of £1 for two years is the amount of R£ for one year. This must evidently be R times the amount of *one pound* for one year;

$\therefore$ the amount of £1 for 2 years $=R^2$,

so	„	1	„	3	„	$=R^3$,
and	„	1	„	n	„	$=R^n$;
$\therefore$	„	P	„	n	„	$=PR^n$;

$$\therefore\ M=PR^n.$$

182. If D be the discount on P due a certain time hence, and I the interest on it during that time,

$$\frac{1}{D}=\frac{1}{P}+\frac{1}{I}.$$

For since I is the discount on $P+I$, we have

$$\text{As } P+I : P :: I : D;$$
$$\therefore (P+1)D=PI;$$
$$\therefore \frac{1}{D}=\frac{P+I}{PI}=\frac{1}{P}+\frac{1}{I}.$$

CHANCES.

183. The subject of Chances consists of two principal parts —*Direct Chances* and *Inverse Chances*,—and in solving any problem it is indispensable to settle first, under which of these two divisions it falls.

The nature of *Direct Chances* will be best understood by a careful study of its main definition, which is as follows :—

If an event can happen in a ways and fail in b ways, **all these ways being equally likely to occur,** *then the chance of its happening is* $\frac{a}{a+b}$ and that of *its failing is* $\frac{b}{a+b}$.

The clause in thick type should be carefully noticed, as from neglecting this, many of the mistakes made in chances occur. The meaning of the above important definition we will exemplify by the solution of three problems.

(i) What is the chance of throwing 7 with a pair of dice?

First we have to find the quantity "a," *i.e.* the number of ways in which 7 can be thrown; each die having 6 faces numbered from 1 to 6, we have the following cases :—

1, 6,
2, 5,
3, 4,
4, 3,
5, 2,
6, 1.

Six in all ; $\therefore a=6$.

Next we have $(a+b)$, the whole number of ways in which the dice can be thrown. This is $6 \times 6 = 36$ (Art. 99);

$$\therefore \text{ by the definition the required chance } = \frac{6}{36} = \frac{1}{6}.$$

Since all the numbers from 1 to 6 are equally likely to be thrown, the condition in thick type is clearly here satisfied.

N.B.—Instead of finding a and b separately it is often better as above to find a, and $(a+b)$.

(ii) m persons sit round in a circle. What is the chance that if three are selected at random no two of those selected are sitting next one another?

The total number of ways in which three persons can be picked out of m is clearly $\frac{m(m-1)(m-2)}{1.2.3}$. This therefore is in this case the value of $(a+b)$.

Next to find "a." Let us first choose one of the three men. We can do this in m ways, and we may now remove the two men on each side of the one selected, as they are excluded by the question. We have $(m-3)$ men left, and we require to pick out of these two men who are not sitting together.

Now the *whole* number of ways in which two men can be picked out of $(m-3)$, is $\frac{(m-3)(m-4)}{1.2}$; and the number of ways in which we can pick two men who *are* sitting together is $(m-3)-1 = m-4$. Hence the number of ways in which we can pick two men *not* sitting together is

$$\frac{(m-3)(m-4)}{1.2} - (m-4) = \frac{(m-4)(m-5)}{1.2}.$$

Thus it would seem that the number of ways in which three persons can be selected, so that no two should sit together, is

$$\frac{m.(m-4)(m-5)}{1.2}.$$

But this is three times too much, for to obtain any particular three men by the above method, say A, B, C, we should first

pick A and then B and C; or we should first pick B, and then C and A; or first C and then A and B, obtaining the same combination in these three different ways;

$$\therefore\ a=\frac{1}{3}\cdot\frac{m.(m-4)(m-5)}{1.2} : \text{and } (a+b)=\frac{m.(m-1)(m-2)}{1.2.3};$$

$$\therefore \text{ the chance required}=\frac{(m-4)(m-5)}{(m-1)(m-2)}.$$

This problem was proposed in the Mathematical Tripos, 1875, and the different points in the solution will repay careful study.

(iii) If three points are taken at random on the circumference of a circle, find the chance of their lying on the same semicircle.

Let n denote the number of points on a unit length of the circumference, so that n is as great as we please and becomes ultimately infinite; therefore the number of points on the circumference of a circle of radius r is $2\pi nr$. Hence $(a+b)$, the number of ways in which three of these points can be picked at random is

$$\frac{2\pi nr(2\pi nr-1)(2\pi nr-2)}{1.2.3}.$$

Next, *in how many ways will any of these triplets be on the same semicircle?*

A semicircle is completely defined by its extremity, since by joining this with the centre we get the diameter. Thus the number of different ways in which we can take a semicircle is $2\pi nr$. There are now $(\pi nr-1)$ points left in our semicircle, and the number of ways in which we can pick out two of them at random is $\frac{(\pi nr-1)(\pi nr-2)}{1.2}$.

Thus $a=\frac{2\pi nr(\pi nr-1)(\pi nr-2)}{1.2}$;

$\therefore$ the chance required is

$$\frac{3.2\pi nr(\pi nr-1)(\pi nr-2)}{2\pi nr(2\pi nr-1)(2\pi nr-2)}=\frac{3\left(1-\frac{1}{\pi nr}\right)\left(1-\frac{2}{\pi nr}\right)}{\left(2-\frac{1}{\pi nr}\right)\left(2-\frac{2}{\pi nr}\right)}=\frac{3}{4}$$

since n is infinite, and $\therefore$ in the limit $\frac{1}{n}$ vanishes as compared with 1.

The above examples have been chosen to illustrate the *unity of method* in the solution of all problems of the direct chances of simple events.

184. *Chances of Compound events.*

If p_1, p_2, are the chances that two events will happen separately (calculated as above), the chance that *one or other* of these two events will happen is (p_1+p_2); for the compound event we are considering happens if *either* of the events happen, and therefore by the fundamental definition of the chance of this is the *sum* of the separate chances.

185. *To find the chance that both events will happen.*

Let "a" denote the number of ways in which the first event may happen, "b" the number of ways in which it may fail, all these ways being equally likely to occur. Let a', b', be similar quantities for the second event. Then (Art. 99) there are $(a+b)(a'+b')$ ways equally likely to occur. In aa' of these both events happen; in bb' both events fail; in $(ab'+a'b)$ one event happens and the other fails;

$\therefore \dfrac{aa'}{(a+b)(a'+b')}$ is the chance that both events happen,

$\dfrac{bb'}{(a+b)(a'+b')}$ " " fail,

and $\dfrac{ab'+a'b}{(a+b)(a'+b')}$. " one happens and the other fails.

Thus, if we denote the chance that the two several events happen by p_1, p_2, we have for the chance of *both* happening p_1p_2; of both failing $(1-p_1)(1-p_2)$; of one happening and the other failing, $p_1(1-p_2)+p_2(1-p_1)$.

The above reasoning is general and *applicable to any number of events.* Thus we see that having determined the chances of

the simple events we may immediately solve any problem in the chances of compound events—taking care to remember that in order to find the chance of *one or other* of "n" simple events will happen, we must **add** the n simple chances, but that to find the chance of *all the n events* happening we must **multiply** the "n" simple chances.

Examples.—(i) A purse contains 10 sovereigns, 4 half-sovereigns, and 8 shillings; and a person draws out one coin at random, what is the chance that it is gold?

Now the required event will happen if he draws either a sovereign or a half-sovereign, and since there are twenty-two coins in all, the chance of his drawing the former is $\frac{10}{22}$, and the chance of his drawing the latter is $\frac{4}{22}$. Hence the chance that he will draw *either the one or the other is* $\frac{14}{22}=\frac{7}{11}$.

(ii) A bag contains m white balls and n black ones, and $(p+q)$ balls are drawn one by one at random. What is the chance that p of them are white and q black?

Suppose the balls are to be drawn in any assigned order, say first p white and then q black ones. Now the chance of drawing one white ball is $\frac{m}{m+n}$; the chance then of drawing a second white ball is $\frac{m-1}{m+n-1}$; the chance of drawing a third white ball is $\frac{m-2}{m+n-2}$, and so on. Similarly for the black ones. Thus the chance of drawing the p white balls and q black ones in any assigned order is the product

$$\frac{m}{m+n}\cdot\frac{m-1}{m+n-1}\cdots\frac{m-p+1}{m+n-p+1}\cdot\frac{n}{m+n-p}\cdots\frac{n-q+1}{m+n-p-q+1}.$$

If we only want p white balls and q black ones in *any* order we must multiply the above chance by $\frac{\underline{|p+q}}{\underline{|p}\,\underline{|q}}$, the number of permutations of $(p+q)$ things, p of which are of one sort and q of another.

186. We may often make considerable use of simple **Analytical Geometry** in solving questions in direct chances. This method we will exemplify by two examples.

(i) A rod of length "a" is broken at random into two parts. Find the chance that the sum of the squares of the broken parts is less than $\frac{3a^2}{4}$.

Let x denote one part and y the other; $\therefore x+y=a$. With the usual notation of Analytical Geometry draw the straight line $x+y=a$, cutting the axes in A and B. Every point on AB has an "x" and a "y," which correspond to a division of the rod, and thus we may represent each division by the corresponding point on AB; with this understanding that all points on AB are equally probable, since the rod is broken at random. Now draw the circle $x^2+y^2=\frac{3a^2}{4}$, and let this cut AB in CD. For all points of AB *inside* the circle, *i.e.* between C and D, the sum of the squares of the corresponding division of the rod is less than $\frac{3a^2}{4}$, and $CD=a-$(as may be easily seen) while $AB=a\sqrt{2}$.

Hence from the fundamental definition the required chance is $\frac{a}{a\sqrt{2}}=\frac{1}{\sqrt{2}}$.

(ii) Two points are taken at random on a given straight line of length a; prove that the chance of their distance apart exceeding a given length $c(<a)$ is $\left(\frac{a-c}{a}\right)^2$.

Let x, y, denote the distances of the two points from one end of the rod; x and y being capable of receiving any value from 0 to ∞. Draw the straight lines $x=a$ and $y=a$. These with the axes of co-ordinates form a square, any point within which obviously gives admissible values of x, y, and all such points are equally likely. But by Hyp. the difference between x and y must exceed c. Hence if we draw the straight lines $x-y=c$ and $y-x=c$, we shall get two triangular spaces outside these

lines and within the square, the sum of whose areas is $(a-c)^2$, and for each point in which the two points on the rod denoted by x and y satisfy the required condition. Thus since the whole area is a^2 the required chance is $\left(\frac{a-c}{a}\right)^2$.

With the notation of the fundamental definition, we see that in the cases just discussed, the "a" and the "b" are both infinite, and that the chance $\frac{a}{a+b}$ takes the form $\frac{\infty}{\infty}$; this fraction we are enabled to evaluate by the method of simple Analytical Geometry. It is essential that the number of quantities should not exceed two. If there be three, say the method of *solid geometry* is involved, but this is too advanced to be more than just mentioned in an elementary work.

187. *Inverse chances.*

In inverse chances an event has taken place, and we have to determine either the chance of its having happened in some assigned way, or that some other event will follow.

Let us suppose that an event has happened which must have resulted from some one of n causes. If these causes are not *a priori* equally probable, let their *a priori* probabilities be denoted by $P_1, P_2 \ldots P_n$. Also *before the event happened* suppose the probability that the event would happen from the first cause is p_1, and so on for the others. *After the event has happened* the probability that it was the result of the first cause must clearly be proportional to the probability that it would happen as the result of that cause, *i.e.* to P_1p_1. Suppose it to be cP_1p_1. Similarly for the other cases. But it *must* have happened from some one of the n causes; and thus the sum of the probabilities of all the causes must be certainty, *i.e.* unity.

$$c(P_1p_1+ \ldots P_np_n)=1; \quad \therefore c=\frac{1}{\Sigma(P_1p_1)};$$

$\therefore$ the probability of the first cause is $\frac{P_1p_1}{\Sigma(P_1p_1)}$, and so on.

We are now in a position to find the chance that a second

event will follow; for we multiply the probability of each cause by the probability that the second event will follow from it, and sum up all such products.

Thus we see that in inverse chances it is of the first importance to find all the possible causes from which the given event can have happened. An example will make this plainer.

Ex. A bag contains n balls, each of which is equally likely to be white or black; a white ball is drawn and replaced: find the chance that another drawing will give a white ball.

Here the *a priori* probabilities which we have denoted by $P_1, P_2, \ldots P_n$ are all equal, and therefore divide out in the numerator and denominator. We need not consequently consider these *a priori* chances here at all. Now there are n different cases to consider, since the number of white balls may be any one of the series 1, 2, . . . n, the rest being black in each case. The probability of drawing a white ball (the observed event) in these cases is $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \frac{n}{n}$ the sum of which is $\frac{n(n+1)}{2n} = \frac{n+1}{2}$.

Hence the probability that there is one white ball is $\dfrac{\frac{1}{n}}{\frac{n+1}{2}} = \frac{2}{n(n+1)}$. So the chance that there are two white balls is $\frac{2.2}{n(n+1)}$, and so on. The chance of drawing another white ball in the *first* case is therefore $\frac{2}{n(n+1)} \cdot \frac{1}{n}$; in the second case $\frac{2.2}{n.(n+1)} \cdot \frac{2}{n}$, and so on. Thus the whole chance of drawing a white ball again is

$$\frac{2}{n(n+1)} \left\{ 1^2 + 2^2 + \ldots + n^2 \right\} \frac{1}{n} = \frac{2n+1}{3n}.$$

Examples.

ELEMENTARY.

1. Factorize $(ax+b)^2-(bx+a)^2$, $x^4+a^2x^2+a^4$, x^6-a^6, $32x^5+243y^5$, $(a+b)^2+a+b-c-c^2$, $(x^2+y^2)^2-4x^2y^2$, $x^3+y^3+z^3-3xyz$.

2. Factorize $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

3. Factorize $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

4. Show that $(a+b+c)^2-(b+c)^2-(c+a)^2-(a+b)^2+a^2+b^2+c^2=0$.

5. Factorize $x-a+\sqrt{x}-\sqrt{a}$. What is the value of this expression if $\sqrt{x}+\sqrt{a}=-1$?

6. Factorize, without introducing imaginary quantities, x^2+a^2.

7. Divide $x^{-3}-y^{-3}$ by $x^{-1}-y^{-1}$ and $x^{pq}-y^{pq}$ by x^p-y^p.

8. What is the remainder when $x^4-x^3+x^2-x+1$ is divided by $x-2$?

9. Divide $x-y$ by $\sqrt[4]{x}-\sqrt[4]{y}$.

10. Divide x^3+x^2-x+2 by $1-x+x^2$.

11. Divide $1+bx$ by $1+ax$, writing down the nth term in the quotient. *Result*, nth term $=(-1)^{n-2}a^{n-2}(b-a)x^{n-1}$.

12. Prove that $1-a^{\frac{1}{2}}-a+a^{\frac{3}{2}}$ is always positive if a is real.

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13. Prove that $(y+az)^3+(z+ax)^3+(x+ay)^3-3(y+az)(z+ax)(x+ay)=(1+a^3)(x^3+y^3+z^3-3xyz)$.

14. If $x^2+y^2+z^2=1$, and $ax+by+cz=0$, show that $a(b^2+c^2)yz+b(c^2+a^2)xz+c(a^2+b^2)xy+abc=0$.

15. If $x+y+z=0$, and $xy(a+b)+yz(b+c)+zx(c+a)=0$, prove that $ax^2+by^2+cz^2=0$.

16. If x, y, and z be unequal, and $x^2+y^2+axy=y^2+z^2+ayz$ $=z^2+x^2+azx$, prove that either of these expressions$=$
$$\frac{x^2+y^2+z^2}{2}.$$

17. If $m=2^n$ prove $(x^2-x+1)(x^4-x^2+1)\dots(x^m-x^{\frac{m}{2}}+1)$ $=\dfrac{x^{2m}+x^m+1}{x^2+x+1}$.

18. Prove
$$(ab+cd)(a^2+b^2-c^2-d^2)-(ac+bd)(a^2-b^2+c^2-d^2)$$
$$=(a+d)(b-c)[(a-d)^2+(b+c)^2].$$

G. C. M. AND L. C. M.

1. Find the G.C.M. of $ax^3-(a^2-1)x^2-a^2$ and $x^3-(a^2-1)x-a$.
Result, $x-a$.

2. Find the G.C.M. of a^2-b^2+2b-1 and $a^2-ab+b-1$.
Result, $a-b+1$.

3. If $x-a$ measure x^2+3x+2 and x^2-2x+4, prove that $5a-2=0$.

4. If $x-2$ measure x^3+ax^2+b and x^3+bx+a, prove $2a=3b$.

5. If $\dfrac{a}{b}=\dfrac{c}{d}$ prove that
$$\frac{\text{G.C.M. of } (a+b) \text{ and } (a-b)}{\text{G.C.M. of } (c+d) \text{ and } (c-d)}=\frac{\text{L.C.M. of } (a+b) \text{ and } (a-b)}{\text{L.C.M. of } (c+d) \text{ and } (c-a)}=\sqrt{\frac{ab}{cd}}.$$

6. If 4 be the G.C.M. of $m+n$ and $m-n$, the G.C.M. of m and n is 2 or 4; if the former, $\dfrac{m}{2}$ and $\dfrac{n}{2}$ are both odd; if the latter, then of the numbers $\dfrac{m}{4}$, $\dfrac{n}{4}$ one is odd and the other even.

7. x^3+x+1, ax^3+bx+c, $a^2x^3+b^2x+c^2$, are three quantities. The G.C.M. of the first and second is $x-p$, and that of the first and third is $x-q$. Prove that $(a+b)q=(a+c)p$.

8. If the L.C.M. of a and b be m times that of c and d, and the G.C.M. of a and b is n times that of c and d, prove $\dfrac{ab}{mn}=cd$.

9. $x-2$ is the G.C.M. of $x^2+ax+2b$ and x^2+bx+c. Show that their L.C.M. is $(x+a+2)(x^2+bx+2a)$.

FRACTIONS.

1. Prove that

$$\frac{a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}=(a+b)(b+c)(c+a).$$

2. Prove that

$$\frac{a^6(b^3-c^3)+b^6(c^3-a^3)+c^6(a^3-b^3)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$$
$$=(b^2+bc+c^2)(c^2+ca+a^2)(a^2+ab+b^2).$$

3. Prove that

$$\frac{bc}{a(a^2-b^2)(a^2-c^2)}+\frac{ac}{b(b^2-a^2)(b^2-c^2)}+\frac{ab}{c(c^2-b^2)(c^2-a^2)}=\frac{1}{abc}.$$

4. Reduce $\dfrac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$ to lower terms.

5. Prove that $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}+\dfrac{b^2-(c-a)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}=1.$

6. If $ad=bc$ then $\dfrac{1}{bc}\left(\dfrac{a}{q}+\dfrac{b}{p}+\dfrac{c}{n}+\dfrac{d}{m}\right)=\dfrac{1}{ma}+\dfrac{1}{nb}+\dfrac{1}{pc}+\dfrac{1}{qd}.$

7. Simplify $\dfrac{m^2+mn+n^2}{(m+n)^3}\times\dfrac{m^2-n^2}{m^3-n^3}.$

8. Simplify $\left\{1+\dfrac{2pq}{p^2+q^2}\right\}\div\left\{\dfrac{p^3-q^3}{p-q}-3pq\right\}.$

9. Prove that

$$(n^{\frac{1}{2}}-n^{\frac{1}{4}}+1)(n^{\frac{1}{4}}-n^{\frac{1}{8}}+1)\ldots ad\ inf.=\frac{n+n^{\frac{1}{2}}+1}{3}.$$

10. Given $x+y+z=0$, prove $\dfrac{x^7+y^7+z^7}{7}=\dfrac{x^4+y^4+z^4}{2}.xyz.$

11. Simplify $\dfrac{x^2-x+1}{x^2+x+1}+\dfrac{2x(x-1)^2}{x^4+x^2+1}$ and hence find the value of n terms of the series of which these are the first two terms, the third term being $\dfrac{2x^2(x^2-1)^2}{x^8+x^4+1}$, the fourth $\dfrac{2x^4(x^4-1)^2}{x^{16}+x^8+1}$, and so on.

12. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\dfrac{g}{h}$, show that $\dfrac{ac+ge}{bd+fh}=\dfrac{ae+cg}{bf+dh}$.

13. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}=\ \ldots$ then will

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}+\ldots$$
$$=\frac{(x+y+z+\ldots)^2+(a+b+c+\ldots)^2}{(x+y+z+\ldots)+(a+b+c+\ldots)}.$$

14. If $abc\ \ldots\ hk$ be n quantities, such that

$$\frac{x}{a}=\frac{a}{b}=\frac{b}{c}=\ \ldots\ \frac{h}{k}=\frac{k}{y},\text{ show that}$$

$$\frac{(x+a)(x+b)\ \ldots\ (x+k)}{(y+a)(y+b)\ \ldots\ (y+k)}=\left(\frac{x}{y}\right)^{\frac{n}{2}}.$$

15. Prove that

$$\frac{x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)-4xyz}{x+y+z-xyz}$$
$$=1-xy-yz-zx.$$

16. If $\dfrac{2bxz^2+3az+2b(x-2c)}{3ayz^2+2bz+3a(y-3c)}$ be constant for all values of z, then will $\dfrac{4bc+3a}{2b+9ac}=\dfrac{2bx-4bc}{3ay-9ac}$.

INVOLUTION AND EVOLUTION.

1. Find the expansions of $(a-bx+cx^2)^3$ and $(a-bx)^5$.

2. Find the square roots of the following expressions :—

(i) $1-2x+3x^2-2x^3+x^4$.

(ii) $(a^2x^2+c^2)-2(abx^2+cd)+(b^2x^2+d^2)+2(a-b)(c-d)x$.

(iii) $(x-y)^4-2(x^2+y^2)(x-y)^2+2(x^4+y^4)$. *Result*, x^2+y^2.

(iv) $\sqrt[3]{x^4}+4\sqrt[3]{x^2}+4\sqrt[4]{x}+2\sqrt[12]{x^{11}}+\sqrt{x}+4$.

3. Find the cube roots of the following expressions :—

(i) $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$.

(ii) $\dfrac{1}{x^3}(x^6-3x^5+6x^4-7x^3+6x^2-3x+1)$.

(iii) $(x^2+y^2)^3+(x^2-y^2)^3+6x^2(x^4-y^4)$. *Result*, $2x^2$.

(iv) $x+\sqrt{x^3}+3\left(\sqrt[6]{x^7}+\dfrac{1}{\sqrt[3]{x^{-4}}}\right)$.

SURDS AND IMAGINARY QUANTITIES.

1. Simplify

$$\frac{1}{1+\sqrt{2}+\sqrt{3}}+\frac{1}{1-\sqrt{2}+\sqrt{3}}+\frac{1}{1+\sqrt{2}-\sqrt{3}}+\frac{1}{1-\sqrt{2}-\sqrt{3}}.$$

Result, 2.

2. Simplify

$$\frac{1+\sqrt{-1}(1-\sqrt{2})}{1-\sqrt{-1}(1-\sqrt{2})}+\frac{1-\sqrt{-1}(1-\sqrt{2})}{1+\sqrt{-1}(1-\sqrt{2})}.$$ *Result*, $\sqrt{2}$.

3. Find $\sqrt{8+2\sqrt{15}}$, $\sqrt{30+12\sqrt{6}}$, $\sqrt{21+4\sqrt{5}}$, $\sqrt[3]{10\sqrt{7}+22}$, and $\sqrt[6]{99-70\sqrt{2}}$.

4. Show that $(1+\sqrt{-1})^n+(1+\sqrt{-1})^{n+1}+(1+\sqrt{-1})^{n+2}+(1+\sqrt{-1})^{n+3}=5\sqrt{-1}(1+\sqrt{-1})^n$.

5. Simplify

$$\frac{xy-x^2\sqrt{-1}}{xy-2x^2\sqrt{-1}}\times\frac{y^2-2xy\sqrt{-1}}{y^2+2xy\sqrt{-1}}\times\frac{2x-y\sqrt{-1}}{x-y\sqrt{-1}}\times\frac{2xy+(x^2-y^2)\sqrt{-1}}{\sqrt{-x^2-y^2}}.$$

Result, $\sqrt{x^2+y^2}$.

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6. Prove that

$$e^{\log\sqrt{1+x}}\cdot\sqrt[6]{(1+x)^5}\cdot\left(\frac{1+x}{\sqrt{1-x}}\right)^{\frac{2}{3}}\cdot\left(\frac{1-x^3}{1+x+x^2}\right)^{\frac{1}{3}}=(1+x)^2.$$

7. Show that $1-ab+\sqrt{1+a^2}-a\sqrt{1+b^2}$
$=(1-a+\sqrt{1+a^2})[1-\frac{1}{2}(1-a-\sqrt{1+a^2})(1-b-\sqrt{1+b^2})]$

Hint.—$a=-\frac{1}{2}(1-a+\sqrt{1+a^2})(1-a-\sqrt{1+a^2})$,

and hence simplify the fraction

$$\frac{1-ab+\sqrt{1+a^2}-a\sqrt{1+b^2}}{1-ab+\sqrt{1+b^2}-b\sqrt{1+a^2}}.$$

8. Simplify $\dfrac{a^2+\sqrt{2ax}+x^2+a+x}{(a+x+1)^2-2ax}$.

9. Prove that the product of any number of factors each of which is of the form $a+b\sqrt{-1}$ is of the same form.

10. Express with rational denominators

$$\frac{1}{\sqrt{3}-\sqrt{2}},\ \frac{2\sqrt{7}+7\sqrt{2}}{3\sqrt{7}+7\sqrt{3}},\ \frac{1}{\sqrt{2}+\sqrt[3]{3}},\ \frac{a+b\sqrt{-1}}{c+d\sqrt{-1}}.$$

11. Show that $\dfrac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}=\dfrac{\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}$ and that either of these fractions may be written $(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$.

12. Given $x=\sqrt[4]{a}$ find the value of

$$\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}.$$

13. If $a=\frac{\sqrt{-3}-1}{2}$,

Find the value of $(x-1)(x-a)(x-a^2)$. *Result*, x^3-1.

14. Simplify $\frac{2a^2-2b^2-ab\sqrt{2}}{(2a^2-b^2)\sqrt{2}}\times\frac{b-a\sqrt{2}}{b\sqrt{2}-a}$.

15. Prove that $\frac{x-y}{\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2}}=\sqrt[3]{x}-\sqrt[3]{y}$.

EQUATIONS AND QUESTIONS INVOLVING THEM.

1. $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}$ $\qquad x=\frac{b}{a}(a-b+c)$.

2. $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}=\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}$ $\qquad x=2\frac{1}{2}$.

3. $\frac{x-4}{x-5}-\frac{x-2}{x-3}=\frac{2x-7}{x-4}-\frac{2x-3}{x-2}$ $\qquad x=3\frac{1}{2}$.

4. $\frac{m\sqrt{c^2-x^2}+n(c-x)}{m\sqrt{c^2-x^2}-n(c-x)}=\frac{ma+nb}{ma-nb}$ $\qquad x=c.\frac{a^2-b^2}{a^2+b^2}$ or c.

5. $\frac{\sqrt{x}+\sqrt{1+x}}{\sqrt{x}-\sqrt{1+x}}=2$ $\qquad x=-1\frac{1}{8}$.

6. $\frac{\sqrt{a+\sqrt{a^2+ax}}+\sqrt{a}}{\sqrt{a+\sqrt{a^2+ax}}-\sqrt{a}}=a$ $\qquad x=a\left[\left(\frac{2\sqrt{a}}{a-1}\right)^4-1\right]$.

7. $\frac{(1-x)(1+a^3)}{(1+x)(1-a^3)}=\frac{1-x+x^2}{1+x+x^2}$ $\qquad x=a$.

8. $ax+by+cz=0$, $x+y+z=0$, and $\frac{x}{b^2-c^2}+\frac{y}{c^2-a^2}+\frac{z}{a^2-b^2}$

$=\frac{1}{(b+c)(c+a)(a+b)}$.

Result, $x=\frac{b-c}{(a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)}$.

9. $xyz=d^3$, $xyw=c^3$, $xzw=b^3$, $yzw=a^3$. *Result*, $x=\frac{bcd}{a^2}$.

10. $ax+cy+bz=cx+by+az$

$$=bx+ay+cz=a^3+b^3+c^3-3abc,$$

Result, $x=y=z=a^2+b^2+c^2-ab-bc-ca$.

11. $x(y+z)=a$, $y(z+x)=b$, $z(x+y)=c$.

$$x=\sqrt{\frac{(a+c-b)(a+b-c)}{2(b+c-a)}}.$$

12. $ax+by+cz+dw=e$, $a^3x+b^3y+c^3z+d^3w=e^3$,

$a^2x+b^2y+c^2z+d^2w=e^2$, $a^4x+b^4y+c^4z+d^4w=e^4$,

$$x=\frac{e(e-b)(e-c)(e-d)}{a(a-b)(a-c)(a-d)}.$$

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13. If $a^2+a\left(xy+\frac{1}{xy}\right)=\frac{x}{y}+\frac{y}{x}+1$,

and $a^2+a\left(xz+\frac{1}{xz}\right)=\frac{x}{z}+\frac{z}{x}+1$,

show that $a^2+a\left(yz+\frac{1}{yz}\right)=\frac{y}{z}+\frac{z}{y}+1$, provided y and z be unequal.

14. $(x^2+y^2)\frac{x}{y}=\frac{100}{3}$, $(x^2-y^2)\frac{y}{x}=\frac{21}{4}$ $\quad x=\pm 4$ or etc. $y=\pm 3$ or etc.

15. If $\frac{x}{a}+\frac{y}{b}=1$, and $\frac{x^2}{a}+\frac{y^2}{b}=\frac{a^3+b^3}{(a+b)^2}$, show that

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=\frac{a^n+b^n}{(a+b)^n}.$$

16. If $ax-by=1$, and $ax^2-by^2=\frac{1}{a-b}$, prove that $ax^n-by^n=(a-b)^{1-n}$.

17. If $ax+by=2$, and $ax^2+by^2=\frac{a+b}{ab}$, prove that

$$bx^n+ay^n=\frac{a^{n+1}+b^{n+1}}{(ab)^n}.$$

18. If $\dfrac{x}{y+z}=a$, $\dfrac{y}{z+x}=b$, $\dfrac{z}{x+y}=c$, find the relation between a, b, and c, and prove that

$$\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ca)}=\frac{z^2}{c(1-ab)}.$$

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19. $x+y=\dfrac{5}{6}$, $\dfrac{1}{x^2}+\dfrac{1}{xy}+\dfrac{1}{y^2}=\dfrac{61}{36}$, $x=\dfrac{3}{2}$ or $-\dfrac{2}{3}$ or etc.

$y=-\dfrac{2}{3}$ or $\dfrac{3}{2}$ or etc.

20. $\dfrac{a(c-d)}{x+a}+\dfrac{d(a-b)}{x+d}=\dfrac{b(c-d)}{x+b}+\dfrac{c(a-b)}{x+c}$,

$x=0$ or $\dfrac{cd-ab}{a+b-c-d}$.

21. $x^2+y(x+1)=4$, $y^2+x(y+1)=2$.

Hint.—Find $(x+y)=2$ or -3.

22. $\sqrt{x^2+3x+5}+\sqrt{x^2+3x+12}=7$, $x=1$ or -4.

23. $x^2-y^2=a^2$, $x^2+xy=b^2$, $x=\pm\dfrac{b^2}{\sqrt{2b^2-a^2}}$.

24. $mnx^2+(n^2-m^2)x-mn=0$, $x=\dfrac{m}{n}$ or $-\dfrac{n}{m}$.

25. $(x+a)^2-2b(y+b)=xy-ab=(y+b)^2-2a(x+a)$

$x=-\dfrac{a+4b}{3}$ $y=-\dfrac{b+4a}{3}$.

26. Is $(b+c-a-x)^2(b-c)(a-x)$
$+(c+a-b-x)^2(c-a)(b-x)+(a+b-c-x)^2(a-b)(c-x)=0$
an identity or an equation? An identity.

27. $xy+xz+z^2=103$, $yz+xy+x^2=64$, $zx+yz+y^2=89$,

$x=3$, $y=5$, $z=8$, or etc.

28. $2\sqrt[4]{x+1}-3\sqrt{x+1}+(x+1)=0$. Show that $x=0$ or -1, and account for the appearance of the rejected value 15.

29. $xy+xz-x^2=a^2,\qquad yz+xy-y^2=b^2,\qquad xz+yz-z^2=c^2.$

Hint.—Prove $xy=\dfrac{a^2b^2}{a^2+b^2-c^2}$.

30. $x^2+xy+y^2=c^2,\qquad x^2+xz+z^2=b^2,\qquad y^2+yz+z^2=a^2.$
Prove $xy+yz+zx=\sqrt{\{\frac{1}{3}(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)\}}$ and solve the equations.

Hint.—Assume $xy+yz+zx=A$, and $x^2+y^2+z^2=B$, and obtain two equations

$$(A+2B)=a^2+b^2+c^2, \text{ and}$$
$$(B+2A)(B-A)=a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2,$$

whence A and B are found.

31. $\sqrt{x^2+ax+a^2}+\sqrt{x^2-ax+a^2}=\sqrt{2(a^2-b^2)}$,

$$x=\sqrt{\frac{b^4-a^4}{a^2-2b^2}}.$$

32. $\sqrt{2x^2-4x+1}+\sqrt{x^2-5x+2}=\sqrt{2x^2-2x+3}$
$+\sqrt{x^2-3x+4},\quad x=-1 \text{ or } \dfrac{\pm\sqrt{5}-1}{2}.$

33. From the equality of three fractions, each of which is of the form $\dfrac{ax+by+cz+d}{l}$, obtain equations of the form $\dfrac{x+f}{p}=\dfrac{y+g}{q}=\dfrac{z+h}{r}$.

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34. If the two values of $\dfrac{x}{y}$ found from $ax^2+2hxy+by^2=0$ be $\dfrac{x_1}{y_1}$, $\dfrac{x_2}{y_2}$, prove that $\dfrac{x_1x_2}{b}=\dfrac{y_1y_2}{a}=\dfrac{x_1y_2+x_2y_1}{-2h}$.

35. x_1x_2 are the roots of $x^2+2px+q=0$, y_1y_2 the corresponding values of y found from $xy+py+bx+a=0$, show that $x_1y_2+x_2y_1=2a$.

36. If $x_1 x_2$ are the roots of $ax^2+bx+c=0$, form the equations, (i) with roots $\frac{1}{x_1}$ and $\frac{1}{x_2}$, (ii) with roots x_1+x_2 and x_1-x_2.

37. $x^2+p_1x+q_1=0$, $x^2+p_2x+q_2=0$, $x^2+p_3x+q_3=0$, are three equations, each pair of which have one root common, show that $(p_2+p_3-p_1)\left(\frac{p_2}{q_2}+\frac{p_3}{q_3}-\frac{p_1}{q_1}\right)=4$.

38. $x^3+px^2+qx+r=0$ and $x^3+rx^2+px+q=0$ have two roots common, show that their sum is $\frac{p-q}{p-r}$, and their product $\frac{r-q}{p-r}$.

39. $x^4+px^3+qx^2+rx+1=0$ and $x^4+rx^3+qx^2+px+1=0$ have a common root and the symbols are all positive, prove that $p+r=q+2$.

40. No real value of x can make the sign of $ax^2+2bx+c$ differ from that of a if $ac-b^2$ be positive.

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41. If for all integral values of n $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, then will $\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{(a+b+c)^{2n+1}}$.

42. Eliminate l, m, n between

$$la+mb+nc=0, \quad lx+my+nz=0, \quad lu+mv+nw=0.$$

43. Eliminate x, y, z between

$$(x+y)^2=4c^2xy, \quad (y+z)^2=4a^2yz, \quad (z+x)^2=4b^2zx.$$

Result, $a^2+b^2+c^2\pm 2abc=1$.

44. If $x=by+cz$, $y=ax+cz$, $z=ax+by$, show that

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2.$$

45. Eliminate x from

$$\frac{a-x}{1+y}=\frac{b}{1+x}=\frac{a+x}{x(1+y)}.$$

46. Eliminate x, y, z, between $x+z=y$, $(a+b)x+(b+c)z=(a+c)y$, $abx+bcz=1+acy$, $(b-c)^2xy=3z\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

Result, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{3}$.

SERIES.

1. Find the sum to n terms of the following series :—

(i) $1+2+3+\ldots$ (ii) $3+5+7+\ldots$

(iii) $\frac{x-y}{x+y}+1+\frac{x+3y}{x+y}+\ldots$ (iv) $\frac{x}{y}+1+\frac{y}{x}+\ldots$

(v) $x^p+x^{p+q}+x^{p+2q}+\ldots$ (vi) $\sqrt{2}+2+\sqrt{2^3}+\ldots$

(vii) $1.2.3+2.3.4+3.4.5.+\ldots$

(viii) $3.5.7+5.7.9+7.9.11+\ldots$

2. Find the sum to n terms and *ad inf.* of

(i) $\frac{2}{3}+\frac{1}{3}+\frac{1}{6}+\ldots$ (ii) $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\ldots$

(iii) $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots$ (iv) $\frac{1}{3.5.7}+\frac{1}{7.9.11}+\frac{1}{11.13.15}+\ldots$

3. Insert 3 A.M. between 3 and 4, and 3 G.M. between 2 and 32.

4. The nth term of a series is $5n-1$: write down the first three terms and find the sum of p terms.

5. The sum of n terms of a certain series is n^2, find the series and the nth term.

6. p, q, r are the nth, $2n$th, $3n$th terms respectively of a G.P. Prove $pr=q^2$.

7. If a, b, c are in H.P., then

$$b^2(a-c)^2=2[(b-a)^2c^2+(c-b)^2a^2].$$

8. The pth term of an A.P. is $\frac{1}{q}$ and the qth term $=\frac{1}{p}\cdot$ Show that the sum of pq terms is $\frac{pq+1}{2}$.

9. Sum to n terms (i) $1^2+4^2+7^2+\ldots$

(ii) $1.2^2.3+2.3^2.4+\ldots$ *Hint.* nth term $=\frac{1}{2}\{(n-1)n(n+1)(n+2)+n(n+1)(n+2)(n+3)\}$.

---

10. a, b, c are the pth, qth, rth terms respectively of (i) an A.P., (ii) a G.P., (iii) an H.P. Show that in the first case $(q-r)a+(r-p)b+(p-q)c=0$; in the second case $a^{q-r}b^{r-p}c^{p-q}=1$; and in the third case $\frac{q-r}{a}+\frac{r-p}{b}+\frac{p-q}{c}=0$.

11. The Arith. and Harm. means between two quantities are A_1, H_1 respectively; and those between two other quantities are A_2, H_2. Show that $A_1H_2+A_2H_1>2H_1H_2$.

12. Show that the sum of the terms in the nth bracket of $1+(3+5)+(7+9+11)+(13+15+17+19)+\ .\ .\ .$ is n^3, and *hence* find the sum of the cubes of the first n neutral numbers.

13. a^2, b^2, c^2, are in A.P., show that $b+c$, $c+a$, $a+b$, are in H.P.

14. S_1 is the sum of an ordinary G.S. of $3n$ terms; S_2 the sum of the same series with alternate signs. The first three terms of the first series are added, and this sum is multiplied by the sum of the first three terms of the second series; the sum of the second triplet of the first series by the sum of the second triplet of the second series; and so on, thus forming a third series of n terms, the sum of which is S_3. Show that

$\frac{S_3}{S_1S_2}=\frac{1-r^6}{1+r^6}\cdot\frac{1-r^{6n}}{(1-r^{3n})^2}$ or $\frac{1-r^6}{1+r^6}\cdot\frac{1+r^{6n}}{1-r^{6n}}$, according as n is even or odd.

15. $M_1\ M_2\ .\ .\ .\ M_n$ are the sums of m consecutive terms of a G.S. beginning with the first, second . . . nth respectively, and $N_1\ N_2\ .\ .\ .\ N_m$ of n consecutive terms of the same series

beginning with the first, second . . . mth respectively. Show that $M_1 + M_2 + \ldots + M_n = N_1 + N_2 + \ldots + N_m$.

16. Between each of $m+1$ pairs of quantities (x, y) $(x, 2y)$ $(x, 4y)$, etc., are inserted m geom. means, and M_1 M_2 M_3 . . . are the nth means respectively: prove that

$$\frac{M_1}{M_2}+\frac{M_2}{M_3}+ \ldots = m \times 2^{-\frac{m}{m+1}}.$$

17. S_1, S_2, S_3 . . . are the sums to n terms of n geometric series, each having unity for the first term, and the common ratios being 1, 2, 3 . . . show that

$$S_1+S_2+2S_3+3S_4+ \ldots +(n-1)S_n = 1^n+2^n+3^n+ \ldots +n^n.$$

18. If the common difference of a series in A.P. be double the first term, show that the quotient obtained by dividing the sum of any number of terms by the first term is a perfect square.

19. If S denote the sum of $2+5x+8x^2+11x^3+$ . . . to n terms, and s that of $1+x+x^2+ \ldots$ to $(n-1)$ terms, show that

$$1+(3n-1)x^n = 3(1+xs)-S(1-x).$$

20. If a, b, c are in H.P., then a, $a-c$, $a-b$, and also c, $c-a$, $c-b$, are in H.P.

21. If the mth term of an H.P. be n, and the nth term m, then will the rth term be $\frac{mn}{r}$.

22. The sum of p, q, r terms of the series of the same A.P. is P, Q, R respectively: show that

$$\frac{P}{p}(r-q)+\frac{Q}{q}(p-r)+\frac{R}{r}(q-p)=0.$$

RATIO, PROPORTION, AND VARIATION.

1. If $\frac{a_0}{a_1}=\frac{a_1}{a_2}=\frac{a_2}{a_3}= \ldots =\frac{a_{r-1}}{a_r}$ then each of these is equal to

$$\frac{a_0^{\frac{1}{r}}+a_1^{\frac{1}{r-1}}+a_2^{\frac{1}{r-2}}+ \ldots +a_{r-1}}{a_r^{\frac{1}{r}}+a_r^{\frac{1}{r-1}}+a_r^{\frac{1}{r-2}}+ \ldots +a_r}.$$

2. Show that the sum of the greatest and least of four proportionals exceeds the sum of the other two.

3. If x be to y in the duplicate ratio of a to b, and a to b in the subduplicate ratio of $a+x$ to $a-y$, show that

$$x+y:x-y::x+a:x.$$

4. If $x, y, z, \ldots$ are not proportional to $a, b, c, \ldots$ show that the ratio of $x+y+z+\ldots$ to $a+b+c+\ldots$ lies between the greatest and least of the ratios $x:a$, $y:b$, ...

5. If each of two quantities vary as a third, show that the geometric and harmonic means between them also vary as the third.

6. If $y \propto \sqrt{a^2-x^2}$ and $=b$ when x is 0, find the equation between x and y.

7. If $\sqrt{y} \propto \sqrt{h}-\sqrt{x}$ and $=\sqrt{k}$ when x is 0, show that $\sqrt{\frac{x}{h}}+\sqrt{\frac{y}{k}}=1$.

8. $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$; prove that a, b, c, d are proportionals.

9. If $a, b, c, d, \ldots$ are quantities such that $x \propto$ as any *one*, when all the rest are constant, then will $x \propto abcd \ldots$ when they all vary.

10. If x vary as $\frac{1}{a}$, a as $\frac{1}{b^2}$, b as $\frac{1}{c^3}$ ... h as $\frac{1}{k^{2n}}$ and $k=1$ when $x=s^{\underline{|2n}}$, n being finite, show that $x=(ks)^{\underline{|2n}}$. Why is it specified that n is finite?

11. A patient would recover from illness by taking p grains of a certain medicine daily for nine days, but prefers taking p homœopathic globules daily, each of which contains $\frac{1}{p^q}$ grains. He then recovers in pq days. Compare the efficacy of his imagination with that of the medicine. *Result*, $p^{q-1}-1:1$.

12. One root of each of two quadratics varies as a, and the other as b, the sum of the roots is 8 if $a=1$ and $b=2$, and the product of the roots is 12 if $a=2$ and $b=1$. What are the equations?

Result, $(x-2a)(x-3b)=0$ or $(x-6a)(x-b)=0$.

PERMUTATIONS AND COMBINATIONS.

1. Prove, independently of the Binomial Theorem, that the *total* number of combinations of n things is 2^n-1.

2. If there be $(n+1)$ different sets of n things, of which there are two kinds, and the first set consist all of one kind, the second set of $n-1$ of the first kind and one of the second kind, the third set of $n-2$ of the first kind and 2 of the second kind, and so on: show that the sum of the permutations which can be formed by taking all in each set together is 2^n.

3. Find n from the equations (i) ${}^nP_2=5.{}^{\frac{n}{2}}P_2$.

(ii) ${}^nC_{n-2}:{}^{2n}C_{2n-3}::3:44$.

4. In how many combinations of n things r together will at least 2 of any given 3 occur? *Result*, $3.{}^{n-2}C_{r-2}+{}^{n-3}C_{r-3}$.

5. There are m quantities a, b, c, . . . and n other quantities x, y, z, . . . How many combinations can be formed with r of the former and s of the latter, where each combination contains a but not x?

6. If $p_2\,p_3$ . . . p_n denote the number of permutations which can be formed of n things 2, 3, . . . n together respectively; and their product be denoted by P, prove that

$$P=p_3.p_{n-1}[(p_3-p_2)(p_4-p_3)\;\ldots\;(p_{n-1}-p_{n-2})].$$

7. If the number of combinations of n things $p+q$ together be equal to the number of combinations of n things $p-q$ together, show that $n=2p$.

BINOMIAL THEOREM.

1. Find the middle term of $(a-x)^{2n}$ in the form of $\dfrac{1.3.5\ldots(2n-1)}{\lfloor n}(-2ax)^n$.

2. Show that no odd powers of x occur in the product $(1-x+x^2-x^3+\ldots \textit{ad inf.})(1+x+x^2+x^3+\ldots \textit{ad inf.})$

3. Find the sum of the squares of the coefficients in the expansion of $(1+x)^n$ where n is a positive integer.

Hint.—Let $(1+x)^n=f(x)$ and $\therefore \left(1+\frac{1}{x}\right)^n=f\left(\frac{1}{x}\right)$. Multiply these two together, etc.

4. The coefficient of x^{2n-1} in the expansion of $\left(x-\frac{1}{x}\right)^{4n+1}$ is $\frac{\underline{|4n+1}}{\underline{|3n}\,\underline{|n+1}}(-1)^{n+1}$.

5. Prove that

$$x^{2n}\left[1+\frac{n}{1}\cdot\frac{a^2}{a^2+x^2}+\frac{n(n+1)}{1.2}\left(\frac{a^2}{a^2+x^2}\right)^2+\ldots\right]=(x^2+a^2)^n.$$

6. If $f(m)=(1+x)^m$ prove that $\frac{f(2n+1)-f(1)}{x\{f(1)+f(0)\}}$

$$=f(0)f(1)+f(1)f(2)+f(2)f(3)+\;.\;.\;.\;+f(n-1)f(n).$$

7. Find the term in the expansion of $(1+x)^{11}$ whose coefficient is double the preceding one. *Result*, the 5th.

8. Show that

$$\frac{4.7\ldots(3r+1)}{\underline{|r}}+\frac{4.7\ldots(3r-2)}{\underline{|r-1}}\,\frac{5}{1}+\frac{4.7\ldots(3r-5)}{\underline{|r-2}}\cdot\frac{5.8}{1.2}$$
$$+\;.\;.\;.\;+\frac{5.8\ldots(3r+2)}{\underline{|r}}=3^r\frac{(r+1)(r+2)}{2}.$$

Hint.—Expand $(1-x)^{-\frac{4}{3}}$ and $(1-x)^{-\frac{5}{3}}$ etc.

9. If m n be the qth terms of the expansions of $(1-x)^{-\frac{1}{2}}$ and $(1-x)^{-\frac{3}{2}}$ respectively, show that $n=(2q-1)m$.

10. If r is numerically greater than $\frac{n-1}{2}$, n not being a positive integer, and R denote the $(r+1)$th term in the expansion of $(1+x)^n$, show that the error arising from taking the sum of the first r terms to represent the sum *ad inf.* is $<\frac{R(1+r)}{1+r(1+x)-nx}$, x being a proper fraction.

11. Show that

$^{2n}C_n + {^{2n}C_{n-1}}{^nC_1} + {^{2n}C_{n-2}}{^nC_2} + \dots + {^{2n}C_1}{^nC_{n-1}} + 1 = {^{3n}C_{2n}}$.

12. The coefficients in the expansion of $(1+x)^n$ where n is not a positive integer are $p_1 p_2 p_3$ . . . show that

$$\text{(i)}\ p_1 + p_1p_2 + p_2p_3 + p_3p_4 + \dots ad\ inf. = \frac{\lfloor 2n}{\lfloor n+1 \lfloor n-1}.$$

$$\text{(ii)}\ p_2 + p_1p_3 + p_2p_4 + p_3p_5 + \dots = \frac{\lfloor 2n}{\lfloor n+2 \lfloor n-2}.$$

$$\text{(iii)}\ p_r + p_1p_{r+1} + p_2p_{r+2} + p_3p_{r+3} \dots = \frac{\lfloor 2n}{\lfloor n+r \lfloor n-r}.$$

13. If N be the integral part of $(7+4\sqrt{3})^n$, show that $N-2^{2n+1}+1$ is a multiple of 6.

14. If N be the integral part of $(2+\sqrt{5})^n$, show that N and 2^{n+1} end with the same digit if n be odd, but with two consecutive digits if n be even.

15. Show that the integral part of $(\sqrt{7}+\sqrt{6})^{2n}$ is odd.

16. Find the term involving x^8 in the expansion o $(x-x^2-x^3)^4$.

17. How many terms are there in the expansion of $(a+b+c+d)^6$.

18. Find the term involving x^2 in the expansion of $(a+bx+cx^2)$.

INDETERMINATE COEFFICIENTS.

1. If $(1+xv)(1+x^2v)(1+x^3v)$ . . . $(1+x^pv) = 1 + A_1v + A_2v^2 + \dots$ show that $A_r = A_{r-1} \times \frac{x^r - x^{p+1}}{1-x^r}$, and hence prove that the given product

$$= 1 + \frac{1-x^p}{1-x}xv + \frac{(1-x^p)(1-x^{p-1})}{(1-x)(1-x^2)}x^3v^2 + \dots$$

2. From the preceding theorem deduce the Binomial Theorem for a positive integral index, and explain why this limitation is here necessary.

3. Show by partial fractions that the coefficient of x^n in the expansion of $\frac{2a}{a^2-x^2}$ is $\frac{1}{a^{n+1}}\{1+(-1)^n\}$.

4. Separate into partial fractions $\frac{x}{(x^2+1)(x-1)^2}$ and show that in the expansion the coefficient of x^{2n} is

$$\tfrac{1}{2}\{(2n+1)+(-1)^n\}.$$

5. Separate into partial fractions $\frac{a}{x^3-a^3}$ and $\frac{x}{x^3+a^3}$.

6. Revert the series

(i) $y=x-\frac{x^3}{3}+\frac{x^5}{5}-\dots$ *Result*, $x=y+\frac{y^3}{3}+\frac{2y^5}{15}+\dots$

(ii) $y=x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\dots$ *Result*, $x=y+\frac{y^3}{6}+\frac{3y^5}{40}+\dots$

(iii) $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$ *Result*, $x=y+\frac{y^2}{2}+\frac{y^3}{6}+\dots$

7. Separate into partial fractions

$$\frac{1}{(x-1)^2(x+1)^3},\ \frac{x}{(x+1)(x+2)^2},\ \text{and}\ \frac{2x+3}{(x+2)^3(x-4)^2}.$$

8. Show that $1^5+2^5+3^5+\dots+n^5=\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}$.

9. If a_{n-1}, a_n, a_{n+1}, be three consecutive coefficients in the expansion of $\frac{1+2x+3x^2}{1-2x-3x^2}$, prove that $a_{n+1}=2a_n+3a_{n-1}$.

CONTINUED FRACTIONS.

1. If $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$, be any three consecutive convergents to $\sqrt{a^2+1}$, show that $\frac{p_{n+1}-p_{n-1}}{p_n}=2a$.

2. Find the first three convergents to the greater root of $x^2-2x-2=0$. *Results*, $1, \frac{3}{2}, \frac{4}{3}$.

3. $\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\ldots$ is a continued fraction, show that b times this fraction, together with its square, is equal to $\frac{b}{a}$.

4. If $\frac{p_n}{q_n}$ denote the nth convergent to $\sqrt{a^2+1}$, show by the method of induction that

$$\frac{p_n}{q_n}=\sqrt{a^2+1}\cdot\frac{(a+\sqrt{a^2+1})^n+(a-\sqrt{a^2+1})^n}{(a+\sqrt{a^2+1})^n-(a-\sqrt{a^2+1})^n}.$$

5. Show that the partial quotients of

(i) $\sqrt{5}$ are 2, 4, 4, 4, 4, . . .
(ii) $\sqrt{6}$,, 2, 2, 4, 2, 4, . . .
(iii) $\sqrt{7}$,, 2, 1, 1, 1, 4, . . .

and hence obtain the first five convergents for each root.

6. What convergent not having more than three figures in the numerator, approaches nearest to the value of $\sqrt{10}$?

Result, the 4th.

7. If $\frac{n_r}{d_r}$ be the rth convergent to a given fraction, and q_r the rth partial quotient, show that

$$q_3+q_4+q_5+\ldots=\frac{n_3+d_3}{n_2+d_2}+\frac{n_4+d_4}{n_3+d_3}+\frac{n_5+d_5}{n_4+d_4}+\ldots$$
$$-\left(\frac{n_1+d_1}{n_2+d_2}+\frac{n_2+d_2}{n_3+d_3}+\frac{n_3+d_3}{n_4+d_4}+\ldots\right).$$

8. Show that $\frac{15}{14}$ differs from $\frac{14863}{13863}$ by less than $\frac{1}{190}$.

9. Each partial quotient from the third inclusive being p, and s being the sum *ad inf.* of the numerators $n_1\, n_2$ . . . of the first, second, . . . convergents: show that $p=\frac{n_1+n_2}{n_1-s}$.

10. N_n is the numerator of the nth convergent to $\frac{a_1}{b_1+}\frac{a_2}{b_2+}$: show that $N_n=b_nN_{n-1}+a_nN_{n-2}$.

If the given fraction were

$$\frac{a_1}{b_1}-\frac{a_2}{b_2-}\ldots \text{ then } N_n=b_nN_{n-1}-a_nN_{n-2}.$$

11. Show that the nth convergent to $\frac{1}{2-}\ \frac{1}{2-}\ \frac{1}{2-}$ is $\frac{n}{n+1}$.

12. Show that

$$\left(a+\frac{1}{2a+}\ \frac{1}{2a+}\ldots\right)^4-\left(a-\frac{1}{2a-}\ \frac{1}{2a-}\ldots\right)^4=4a^2.$$

13. If $x=$ the infinite continued fraction $\frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{c+}\ \frac{1}{a+}\ldots$ and $y=$ the infinite continued fraction $c+\frac{1}{b+}\ \frac{1}{a+}\ \frac{1}{c+}\ldots$ x_1, x_2 being the roots of the quadratic obtained from the first, and y_1, y_2 the roots of the quadratic obtained from the second, show that $x_1+x_2=-(y_1+y_2)$ and $x_1x_2=y_1y_2$. Hence prove

$$xy=\frac{1+bc}{1+ab}.$$

14. Show that the ratio of the infinite continued fraction $a+\frac{1}{a+}\ \frac{1}{a+}\ldots$ to the infinite continued fraction $b+\frac{1}{b+}\ \frac{1}{b+}$ is $\frac{a+\sqrt{a^2+4}}{b+\sqrt{b^2+4}}$.

15. Two scales are placed together, with their zero points coinciding, and 123 divisions of the one are just equal to 100 divisions of the other: show that the divisions which most nearly coincide without actually doing so are $123m+16$ on the one, and $100m+13$ on the other, where m may be 0 or any positive integer.

16. Mercury makes its revolution in 88 days and Venus in

252 days: show that the former will have made very nearly 23 revolutions while the latter has made 9.

17. Prove that $\left(\frac{1}{a+}\,\frac{1}{b+}\ldots\right)\times\left(b+\frac{1}{a+}\,\frac{1}{b+}\ldots\right)=\frac{b}{a}\cdot$

INEQUALITIES.

If a, b, c, etc., denote unequal quantities, show that

1. $(a^4+b^4)(b^4+c^4)(c^4+a^4)>8a^4b^4c^4$.

2. $$\frac{(a^4+6a^2b^2+b^4)(b^4+6b^2c^2+c^4)(c^4+6c^2a^2+a^4)}{2^4}$$
$$>(2abc)^2(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

3. $a^2+b^2+c^2>ab+bc+ca$.

4. $a^3+b^3+c^3>3abc$.

5. $a^4+b^4+c^4+d^4>4abcd$.

6. $\dfrac{4a^4+3b^3}{7a^2b}>\sqrt[7]{a^2b}$.

7. $\sqrt[7]{\underline{|7}}<4$.

8. $(1+a)^4>1+4a(1+a)^{\frac{3}{2}}$. *Hint.—Consider the first four powers of* $(1+a)$.

9. $\left\{1-\dfrac{2mb}{n(a+b)}\right\}^n>\left(\dfrac{a-b}{a+b}\right)^m$. *Hint.*—Consider m quantities each equal to $(a-b)$ and $(n-m)$ each equal to $(a+b)$.

10. $\left\{\dfrac{a(a+1)^2}{4}\right\}^n>(\underline{|n})^3$.

11. $a^3+7-3(b^2+c^2)>3a^2-ab^2-ac^2$, a being >3.

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12. a and b are the first and last terms of n terms in H.P.; show that the sum of all the terms is less than $(a+b)\,\dfrac{n}{2}\cdot$

13. If $a+b+c=1$, then will $\left(\dfrac{1}{a}-1\right)\left(\dfrac{1}{b}-1\right)\left(\dfrac{1}{c}-1\right)>8$.

14. If $a+b+c+d+e=0$, then will
$$(a+b+c)(a+b+e)(a+d+e)(c+d+e)>16bcd\sqrt{ae}.$$

NOTATION AND THEORY OF NUMBERS.

1. Express the number 2378469 in the quinary scale.
Result, 1102102334.

2. Express $\frac{3}{7}$ in the senary scale. *Result*, $.\dot{2}\dot{3}$.

3. Multiply 4376 by 6734 in the nonary scale, and verify the result by converting them into the decimal scale, multiplying and reconverting the product.

4. If a number be expressed in the scale of R by the digits a, b, c (c being that in the unit's place), and in the scale of r by the digits inverted, show that the number is $c\left(\frac{Rr-1}{R-1}\right)^2$ it being given that $b^2=4ac$.

5. Show that the difference of two numbers composed of the same digits is divisible by the highest digit in the scale.

6. Prove that 441 and 144 are both square numbers in any scale whose radix is >4, that their difference is divisible by the numbers next less and next greater than the radix, and that one of the quotients thus obtained exceeds the other by 6.

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7. If N be the sum of two squares, prove that so likewise is 2^nN.

8. Prove that if the last p digits of a number are divisible by 2^p so is the number.

9. Prove that (i) $n(n+1)(2n+1)$ is divisible by 6.
(ii) $7^{2n}-50n+1$,, 100.
(iii) $2^{3n}-7n-1$,, 49.

10. If n be any prime number >2 except 7, n^6-1 must be divisible by 56.

11. If N be prime to 40, N^4 must be of the form $5n+1$ or $8n+1$.

12. If m, n be prime numbers, and prime to N, show that $N^{mn(m+n-2)}-1$ is divisible by m^2n^2.

13. If P be the sum of the odd terms, Q of the even terms, in the expansion of $(2+\sqrt{3})^{2m}$, show that P^2-Q^2 is of the form $37n$.

14. If m be a prime number, and prime to N, show that $N^{n^{r+1}-n^r}-1$ is divisible by n^{r+1}.

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15. If n be prime to 3, then either n^2-n-2 or n^2+n-2 is divisible by 18.

16. Show that the *product* of all the divisors of a^pb^q where a and b are prime numbers, is $(a^pb^q)^{\frac{(p+1)(q+1)}{2}}$.

Extend this to the case in which any given number is separated into the product of powers of three prime numbers.

Hence if n be the *number* of divisions of any given number N, and P the *product* of all the divisions, show that $P^2=N^n$.

17. Find the smallest value of x for which x^2+x+41 is not a prime.

18. $a+2b+4c$ is of the form $8p$, and a, b, c are the last three digits in a certain number (a in the unit's place). Show that the number is divisible by 8.

19. From Wilson's Theorem show that if p be a prime >2, then $\left\{\left\lfloor\frac{p-1}{2}\right.\right\}^2+(-1)^{\frac{p-1}{2}}$ is divisible by p.

20. If a and b be prime to each other, show that no two of the quantities $a, 2a, 3a, \ldots (b-1)a$ when divided by b, can give the same remainder.

21. Show that $7^{2n}-1$ is of the form $24m$, and $7^{2n-1}+1$ of the form $8m$.

CONVERGENCE AND DIVERGENCE.

Are the following series convergent or divergent?

(i) $\frac{4}{x^2-4}+\frac{2}{x^2+10x+24}+\frac{2}{x^2+18x+80}+\ldots$ *Result*, Conv.

(ii) $\frac{1}{x(x+1)}+\frac{1}{(x+2)(x+3)}+\frac{1}{(x+4)(x+5)}+\cdots$

Result, Conv.

(iii) $1+\frac{x}{2}+\frac{x^2|\underline{2}}{3^2}+\frac{x^3|\underline{3}}{4^3}+\cdots$ *Result*, Conv. if $x<e$.

(iv) $\frac{1}{3}+\frac{1}{3.4}+\frac{1}{3.4.5}+\cdots$ *Result*, Conv.

(v) $\frac{a}{m+p}+\frac{a^2}{m+2p}+\frac{a^3}{m+3p}+\cdots$ *Result*, Conv. if $a<1$.

(vi) $\frac{m+p}{a}+\frac{m+2p}{a^2}+\frac{m+3p}{a^3}+\cdots$ *Result*, Conv. if $a>1$.

(vii) $1+\frac{a}{2a+1}+\frac{a^2}{3a^2+2}+\frac{a^3}{4a^3+3}+\cdots$ *Result*, Div.

(viii) $1+\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}+2}+\frac{1}{\sqrt{4}+3}+\cdots$ *Result*, Div.

(ix) $1+nx+\frac{n(n-1)}{|\underline{2}}x^2+\cdots$ *Result*, Conv. if $x<1$.

RECURRING SERIES.

Sum *ad infin.* the following series :—

(i) $1+2x+5x^2+12x^3+29x^4+\ .\ .\ .$

(ii) $x+3x^2+10x^3+33x^4+\ .\ .\ .$

(iii) $x+x^2+x^3+3x^4+5x^5+9x^6+\ .\ .\ .$

(iv) $1+2x+3x^2+9x^3+23x^4+\ .\ .\ .$

INDETERMINATE EQUATIONS, CHANCES, etc.

1. A man has to pay a debt of 16 shillings, and he has only half-crowns and florins, show that there is only one way in which he can pay if he gives some of each.

2. What is the simplest way in which A, who has only half-crowns, can pay 14 shillings to B, who has only four penny pieces?

3. A person paid a guinea for eight books, three of one sort and five of another; he forgets the prices paid for each, but remembers that it was a certain number of *shillings*. What were the prices?

4. There are three numbers, the sum of which is 6, and three times the first together with four times the sum of the second and third is 23. Find them.

5. A person owes $P£$ but is allowed to pay by instalments of $A£$ at the *end* of every year, being charged interest for the amount of debts unpaid at the *beginning*. Show that if r be the interest of £1 for one year, and n the number of years which elapse before the debt is paid off, then

$$n=\frac{\log A-\log(A-rP)}{\log(1+r)}.$$

6. Two equal sums of money are placed out at *very small* rates of compound interest. Show that the times in which they reach the same amount are nearly inversely proportional to the rates of interest.

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7. Two points are taken at random on a rod of length a. Show that the chance of their both lying on the same side of the middle point is $\frac{1}{2}$.

8. Two points are taken at random on a rod of length a, find the chance that the sum of their distances from a given end is less than $\frac{a}{2}$. *Result*, $\frac{1}{8}$.

9. What is the chance that the sum of the squares of the distances of the two points in the last question from the given end is $>\frac{a^2}{4}$? *Result*, $\frac{\pi}{16}$.

10. A bag contains 5 coins, which are equally likely, *à priori*, to be sovereigns or shillings, and a person draws out two at random, which are both sovereigns. He replaces these and is allowed to draw out one coin. What is the value of his expectation?

11. A rod of length $6a$ is coloured in the order red, white, blue, $3a$ being red, $2a$ white, and a blue. Two points are taken at random in adjacent parts of the rod, and it is found that their distance apart is $<a$. What is the chance that the points are in the red and white? *Ans.* $\frac{1}{7}$.

MISCELLANEOUS.

1. Show that the coefficient of x^r in the expansion of $\frac{(1-2x)^2}{(1-x)^4}$ is $\frac{(r^2-1)(r-6)}{6}$.

2. Prove that $\log_e 101-\log_e 99=\frac{1}{50}$ nearly.

3. If $\alpha^x\beta^y\gamma^z=a$; $\alpha^y\beta^z\gamma^x=b$; and $\alpha^z\beta^x\gamma^y=c$,

$$\text{show that } x+y+z=\frac{\log abc}{\log \alpha\beta\gamma}, \text{ and hence find } x.$$

4. Prove that $(a+b-c)^3+2(b+c-a)^3+(c+a-b)^3$

$$=2(b+c)^3, \text{ if } \frac{1}{b}+\frac{1}{c}=\frac{4}{a}.$$

5. Find the numerical value of

$$\frac{c}{b}\cdot\frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}} \text{ if } a(b-c)^2-c(b+c)^2=0.$$

Result, 1.

6. If the $(n+1)$ digits of the number N when expressed in the scale of r be $p_0, p_1 \ldots p_n$, show that the number $r^{n+1}-(N+1)$ will have corresponding $(n+1)$ digits $r-(p_0+1)$, $r-(p_1+1), \ldots$ $r-(p_n+1)$.

7. If $x^2+y^2+z^2=xyz+4$, then $(yz-x)^2+(zx-y)^2+(xy-z)^2$ will be equal to $(yz-x)(zx-y)(xy-z)+4$.

8. If $s=a+b+c$, then $(as+bc)(bs+ac)(cs+ab)$

$$=(b+c)^2(c+a)^2(a+b)^2.$$

9. Show that $\frac{x+1}{x}=1+\frac{1}{1+x}+\frac{1}{(1+x)^2}+\cdots$

10. Prove that

$$\sqrt{n^2-2}=\frac{n^2-1}{n}\left[1-\frac{1}{2}\cdot\frac{1}{(n^2-1)^2}-\frac{1}{8}\,\frac{1}{(n^2-1)^4}-\dots\right].$$

11. Prove that the sum of the roots of $\frac{p}{x-a}+\frac{q}{x-b}+\frac{r}{x-c}=0$ is equal to $\frac{p(b+c)+q(c+a)+r(a+b)}{p+q+r}$.

12. Solve $(x-2)^2(x-1)+2(x-2)(x-1)+x-9=0$.

Result, $x=3$ or $\pm\sqrt{-3}$.

13. The A.M. between two quantities exceeds the H.M. by a, and the space of the A.M. exceeds the sum of the squares of the G.M. and H.M. by a^2: show that the quantities are $a(2+\sqrt{2})$ and $a(2-\sqrt{2})$.

14. Solve $\frac{3x-2}{2}+\sqrt{2x^2-5x+3}=\frac{(x+1)^2}{3}$.

Result, 2, or $\frac{1}{2}$, or, etc.

15. Sum $4a+5a^2+6a^3+\ \dots$ to n terms.

16. The nth term of an A.P. is a G.M. between the sum of n terms, and twice the common difference: show that the ratio of the first term to the common difference is $1\pm\sqrt{n}$.

17. Prove $\frac{(x+z)\sqrt{y^2+z^2}-(y+z)\sqrt{z^2+x^2}}{z(x-y)}$

$$-\frac{2(z^2-xy}{(x+z)\sqrt{y^2+z^2}+(y+z)\sqrt{2^2+x^2}}=0.$$

18. Solve $\sqrt[3]{x}-\sqrt[3]{y}=a$; $\sqrt[3]{x-y}=b$.

Result, $x=\frac{1}{8}(a+k)$, $y=\frac{1}{8}(a-k)$, where $k=\sqrt{\frac{4b^3-a^3}{3a}}$.

19. There are sixteen routes from Q to R, including routes through P, eight from R to P, including routes through Q, and eight from P to Q, including routes through R. How many direct routes are there between each pair of the towns?

20. Solve $\frac{x^2-y}{1-xy}=a$; $\frac{y^2-x}{1-xy}=b$.

Result, $x=\frac{a^2-b}{1-ab}$, $y=\frac{b^2-a}{1-ab}$.

21. Solve $x+\frac{1}{y+}\,\frac{1}{a}=a$, and $y+\frac{1}{x+}\,\frac{1}{b}=b$.

Result, $x=\frac{ab-1}{b}$, $y=\frac{ab-1}{a}$.

22. If $(1+x+x^2)^n=1+A_1x+A_2x^2+\ \ldots$ prove that
$1+A_3+A_6+\ldots=A_1+A_4+A_7+\ldots=A_2+A_5+A_8+\ldots=3^{n-1}$.

23. Solve $x^2+y^2+z^2+a^2+b^2+c^2=2ax+2by+2cz$.

24. If $\frac{y}{b}+\frac{c}{z}=1=\frac{z}{c}+\frac{a}{x}$ then will $xyz+abc=0$.

25. If $x^4-4x^3y+6x^2y^2-4xy^3+4y^3z-by^2z^2+4yz^3-z^4=0$, show that either $x=z$ or x, y, z are in A.P.

26. If $\frac{x}{y+z}=a$, $\frac{y}{z+x}=b$, $\frac{z}{x+y}=c$, show that

$$bc(1+a)+ca(1+b)+ab(1+c)=1+abc.$$

27. If $xy+yz+zx=1$, prove that

$$x\sqrt{\frac{(1+y^2)(1+z^2)}{1+x^2}}+y\sqrt{\frac{(1+x^2)(1+z^2)}{1+y^2}}+z\sqrt{\frac{(1+x^2)(1+y^2)}{1+z^2}}=2.$$

28. Prove

$$4(a^2+ab+b^2)^3-27(a+b)^2a^2b^2=(a-b)^2(a+2b)^2(2a+b)^2.$$

29. If $\alpha+\beta+\gamma=1$, and x_1x_2 be the roots of

$$\frac{\alpha}{a-x}+\frac{\beta}{b-x}+\frac{\gamma}{c-x}=0,$$

and y_1, y_2 be the corresponding values of y as given by

$$y=\frac{a^2\alpha}{a-x}+\frac{b^2\beta}{b-x}+\frac{c^2\gamma}{c-x},$$

show that $x_2+y_1=x_1+y_2=a+b+c$.

30. If from the product of n consecutive integers beginning with x, we subtract the product of n consecutive integers beginning with y, the remainder will be divisible by $(x-y)$. Prove this.

31. Find the coefficient of x^r in the product of

$$(1+x+x^2+\ldots ad\ inf.)(1+2x+3x^2+\ldots ad\ inf.)(1+3x+6x^2+\ldots ad\ inf.)$$

32. Prove that the algebraical sum of the coefficients of the first r terms in the expansion of $(1-x)^n$ is

$$\frac{(n-1)(n-2)\ \ldots\ (n-r+1)(-1)^{r-1}}{\lfloor r-1}.$$

33. If the algebraical sum of the coefficients of the first r terms in $(1-x)^n$ be double what it is in $(1-x)^{n+1}$, show that $n+r=1$.

34. Show that the integer next greater than $(3+\sqrt{7})^m$ has a factor 2^{m+1}.

35. Sum *ad inf.* $1+x\log x+\dfrac{x^2(\log x)^2}{\lfloor 2}+\dfrac{x^3(\log x)^3}{\lfloor 3}+\ \ldots$

36. Show that

$$n^{n+1}-n(n-1)^{n+1}+\frac{n(n-1)}{1.2}(n-2)^{n+1}-\ldots=\frac{n}{2}\lfloor n+1.$$

37. If $a_1x+b_1y+c_1z=1$, $a_2x+b_2y+c_2z=1$, $a_3x+b_3y+c_3z=1$ be three equations, such that one is deducible from the other two, show that

$$b_1c_2+b_2c_3+b_3c_1=b_2c_1+b_3c_2+b_1c_3,$$
$$c_1a_2+c_2a_3+c_3a_1=c_2a_1+c_3a_2+c_1a_3,$$
$$\text{and } a_1b_2+a_2b_3+a_3b_1=a_2b_1+a_3b_2+a_1b_3.$$

38. If a_1, a_2, a_3, are in A.P., a_2, a_3, a_4, in G.P., and a_3, a_4, a_5, in H.P., prove that a_1, a_3, a_5, are in G.P.

39. Solve $\dfrac{a}{x}-\dfrac{b}{y}=\dfrac{b}{y}-\dfrac{c}{z}=\dfrac{c}{z}-\dfrac{a}{x}$, $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=1$.

40. Prove that $(x+y)(x^2+y^2)(x^4+y^4)\ \ldots\ (x^{2n}+y^{2n})$

$$=x^{4n-1}+x^{4n-2}y+x^{4n-3}y^2+\ \ldots\ +y^{4n-1}.$$

ERRATA.

Page 3 line 13 for m read n.

„ 9 „ 26 for $(mp \pm np)x$ read $(mp \pm nq)x$.

„ 62 „ 16 after $\dfrac{n-r+1}{r}$ add $\times x$.

„ 64 „ 4 for $1 -$ etc., read $1 +$ etc.

„ 67 „ 8 in numerator of fraction read $(n-1)$ for $(n+1)$.

„ 148 „ 15 for "neutral" read "natural."

„ 149 „ 5 for n read m.

„ 150 „ 25 for "nine" read q.